Hypocoercive Renormalisation

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Abstract

Our previous work has shown that a single Fisher information metric on densities and a single operator decomposition $K = G+J$ suffice to unify reversible quantum dynamics, irreversible Markov and Fokker-Planck flows, and weak field Fisher gravity. The reversible companion paper derives Schrödinger evolution from a Fisher metric and a canonical Poisson bracket on (ρ, S). The entropy geometry paper shows that the same metric data support a metriplectic structure with cost-entropy inequalities, curvature coercivity, and a scalar Fisher gravity sector. The irreversible density paper exhibits detailed balance Markov chains, finite dimensional GKLS generators and Fokker-Planck limits whose dissipative dynamics are all realised by a single Fisher-Dirichlet operator on densities, and proves a finite dimensional hypocoercivity theorem for the resulting generators $K = G + J$.

We now move from classification to renormalisation. Once time is measured by a Fisher entropy clock, the slow sectors of UIH generators fall into simple universality classes. For each conserved current block, and in any Fisher compatible coarse graining, the irreversible flow is governed by a single hypocoercive index $r = \lambda_{\text{hyp}}/\lambda_F \geq 1$, the ratio between the hypocoercive rate of K and the Fisher gap of G. We show that in hydrodynamic scaling the small wavenumber limit of this index, $r(k) \to r_\star$, is purely geometric: it is the minimal average diffusion on any symplectic two plane of the reversible sector J in the Fisher metric. In particular $1 \leq r_\star \leq \kappa(A)$, where $A = -G$ and $\kappa(A)$ is its condition number, with equality cases characterised by how the symplectic planes of J intersect the extremal eigenspaces of A.

We construct a concrete two field hydrodynamic ring where the infrared limit has $r_\star = 3/2$, and show numerically that Fisher compatible block renormalisation preserves this index across scales while deliberately bad coarse grainings do not. We then promote the ring to a weakly nonlinear system and demonstrate that the same index controls the decay of a Fisher norm for small perturbations. On the quantum side we give an explicit single qubit GKLS model whose Bloch generator has exactly the same $K = G + J$ structure and the same universal index $r_\star = 3/2$, and outline a three level multi charge GKLS model where the Fisher metric is the full Bogoliubov-Kubo-Mori geometry. Finally we scan random finite dimensional metriplectic pairs to show that $1 \leq r \ll \kappa(A)$ is a generic feature of UIH generators. Together, these results support the view that once an information metric and entropy clock are fixed, hypocoercive renormalisation is governed by a small set of geometric universality classes.

1 Introduction

The Universal Information Hydrodynamics (UIH) programme studies dynamical systems whose reversible and irreversible parts are both constrained by a single information metric. In the reversible paper [1] we showed that a Fisher information metric on densities, together with a canonical Poisson bracket on (ρ, S), singles out Schrödinger dynamics as the unique reversible hydrodynamics compatible with a small set of continuity and covariance axioms. The entropy geometry paper [2] attaches to the same metric data a metriplectic structure with cost-entropy inequalities and curvature coercivity, and shows that in simple settings a scalar Fisher gravity sector can be coupled to density fields. The irreversible companion paper [3] then exhibits finite dimensional GKLS generators, detailed balance Markov chains and Fokker-Planck limits whose dissipative sectors are all realised by a single Fisher-Dirichlet operator on densities, and proves a finite dimensional hypocoercivity theorem for the resulting generators $K = G + J$.

In that setting the symmetric part G is the Fisher gradient flow generator and the skew part J is the reversible circulation. The operator G is negative semidefinite in the Fisher metric and its kernel encodes conserved quantities such as normalisation and stationary densities. On the mean zero subspace $-G$ is positive definite, with a spectral gap λ_F that sets the characteristic diffusion timescale of the underlying metric. The full generator $K = G + J$ has spectrum lying in the closed left half plane and, under mild conditions, generates a hypocoercive semigroup: although neither G nor J alone is strictly coercive in the natural norm, their combination yields exponential relaxation to equilibrium.

The present paper addresses the following question. Given such a UIH generator $K = G+J$, how does its long time behaviour transform under coarse graining? Put differently, once we commit to an information metric and its associated entropy geometry, do irreversible flows fall into simple universality classes under renormalisation?

A first hint is already present in the finite dimensional hypocoercivity theorem of [3]. On the mean zero subspace, define

$$\lambda_F = \min \sigma(-G), \qquad \lambda_{\text{hyp}} = \min\{-\operatorname{Re} \lambda : \lambda \in \sigma(K)\}.$$

The Fisher gap λ_F sets a canonical timescale: in the purely dissipative dynamics $\dot{x} = Gx$, deviations from equilibrium decay no faster than $e^{-\lambda_F t}$ in the Fisher norm. The full hypocoercive rate λ_{hyp} can be larger when reversible transport mixes slow and fast directions. The ratio

$$r = \frac{\lambda_{\text{hyp}}}{\lambda_F} \geq 1$$

is therefore a natural dimensionless index that compares the true relaxation rate in the entropy geometry to the bare Fisher diffusion scale. It depends on both G and J, but is invariant under similarity transformations and common rescalings $K \mapsto cK$, and so is a good candidate for a universal quantity.

In this paper we show that once an entropy clock is fixed by λ_F, and once coarse grainings are required to respect the Fisher metric, the slow sectors of K do indeed fall into simple classes labelled by such an index. In particular we establish three main points.

First, in hydrodynamic scaling, translation invariant UIH generators decompose in Fourier space into blocks $K(k)$ whose symmetric part scales as $G_k \sim -k^2 D$ and whose skew part scales as $J_k \sim k^\beta J_0$ for some exponent β. For physically natural advective couplings one has $\beta = 1$, so the reversible sector is infrared relevant. We show that in that regime the entropy clock index at wavenumber k,

$$r(k) = \frac{\lambda_{\mathrm{hyp}}(k)}{\lambda_F(k)},$$

has an infrared limit $r(k) \to r_\star(D, J_0)$ as $k \to 0$, and that this limit can be written purely in terms of the Fisher metric and the symplectic geometry of J_0. Concretely, $r_\star(D, J_0)$ is the minimal average diffusion on a symplectic two plane of J_0, normalised by the coldest Fisher eigenvalue.

Second, we define a class of Fisher compatible renormalisation maps that act by orthogonal coarse graining followed by an entropy clock rescaling. For generators with exact conservation laws whose slow blocks are invariant under K, these maps reduce to spectral projections on the slowest eigenspaces of $-G$ and preserve the index r exactly. In translation invariant hydrodynamics they approximate projections onto low wavenumbers and hence preserve the infrared value $r_\star$. In contrast, deliberately bad coarse grainings that scramble the Fisher metric drive the index away from its microscopic value and destroy universality. This is demonstrated concretely on a two field ring model.

Third, we show through explicit examples and random scans that the hypocoercive indices defined in this way behave as claimed. A nonlinear two field ring with weak advective corrections exhibits exponential decay in a Fisher norm at a rate equal to λ_{hyp} of its linearisation, with the ratio to the Fisher gap matching the predicted $r_\star$, even though the full equations are nonlinear. A one qubit GKLS model with anisotropic noise and a simple Hamiltonian has a Bloch generator whose (x, y) sector is a $K = G + J$ block with $r_\star = 3/2$, identical to the two field hydrodynamic ring. A three level GKLS model with multiple conserved charges realises the same structure in the full Bogoliubov-Kubo-Mori metric. Finally, large random ensembles of finite dimensional metriplectic pairs confirm that $1 \le r \le \kappa(A)$ is generic, with typical values well below $\kappa(A)$, and that the UIH bounds are not artefacts of special low dimensional examples.

The paper is organised as follows. Section 2 reviews the finite dimensional UIH structure and introduces the Fisher entropy clock and hypocoercive index. Section 3 analyses hydrodynamic scaling and the two field ring, and identifies the infrared index $r_\star$. Section 4 develops the Fisher symplectic normal form and shows that $r_\star$ is a simple geometric invariant. Section 5 defines Fisher compatible renormalisation maps and relates them to spectral projections on slow modes. Section 6 presents the nonlinear two field ring and its entropy clock. Section 7 constructs the explicit qubit and qutrit GKLS examples. Section 8 reports the random ensemble scans. Section 9 summarises the picture and outlines future directions. Appendix 10 documents the Python scripts used in the numerical experiments.

2 Fisher metrics, entropy clocks and hypocoercive index

We begin by recalling the finite dimensional UIH structure and by fixing a canonical choice of time coordinate, the Fisher entropy clock, that renders the hypocoercive index dimensionless and renormalisation friendly.

2.1 Finite dimensional UIH generators

Let V be a real vector space of dimension n equipped with a symmetric positive definite inner product $\langle x, y\rangle_M = x^\top My$ for some SPD matrix M. In the UIH setting this inner product is the Fisher information metric associated with a family of densities or states, as discussed in [2, 3]. A linear generator $K\colon V \to V$ is said to be Fisher compatible if it can be decomposed as

$$K = G + J,$$

where

$$G^\top M = MG, \qquad J^\top M = -MJ.$$

In other words, G is symmetric and J is skew with respect to the Fisher inner product. In coordinates where $M = I$ this simply means that G is symmetric and J is real skew.

We will always assume that G is negative semidefinite in the Fisher metric, with a one dimensional kernel spanned by a distinguished equilibrium vector $x_\star$ that represents the stationary density or state. In a probability setting one can take $x_\star$ to be the constant vector of ones in a mass conserving basis. We denote by V_0 the orthogonal complement of the equilibrium direction in the Fisher metric,

$$V_0 = \{x \in V : \langle x, x_\star\rangle_M = 0\}.$$

On V_0 the symmetric part satisfies $-G > 0$, so it generates a strictly contracting gradient flow in the Fisher norm, while the full generator $K = G + J$ is assumed to have spectrum lying in the closed left half plane, with no other eigenvalues on the imaginary axis.

This is the finite dimensional UIH setting of [3]. It covers, in particular, the density sector of finite state Markov chains with detailed balance, restriction of GKLS generators to diagonal density matrices in a preferred basis, and finite volume discretisations of Fokker-Planck equations.

2.2 Fisher gap and entropy clock

On V_0 the symmetric part G defines a self adjoint negative operator in the Fisher metric. We write its spectral decomposition as

$$-G = \sum_{i=1}^{n-1} \lambda_i P_i,$$

with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and orthogonal projectors P_i. The smallest positive eigenvalue,

$$\lambda_F = \lambda_1 = \min \sigma(-G|_{V_0}),$$

is the Fisher gap. It sets the slowest decay rate for the pure gradient flow $\dot{x} = Gx$ on the mean zero subspace: no component of x can decay faster than $e^{-\lambda_F t}$ in the Fisher norm under this dynamics.

In many physical examples G is a discretised Laplacian or Dirichlet form. The Fisher gap then encodes the slowest diffusive mode, and its inverse controls equilibration times over the system size. Crucially, λ_F depends only on the symmetric dissipative structure and the information metric, not on any arbitrary microscopic choice of time units.

This suggests using λ_F to define a canonical time coordinate. Suppose the physical time variable is t. We define the entropy clock τ by

$$\tau = \lambda_F t.$$

In these units the pure dissipative dynamics has slowest rate equal to 1. More precisely, for the flow $\dot{x} = Gx$ on V_0, the Fisher norm $\|x\|_M^2 = \langle x, x\rangle_M$ obeys

$$\|x(\tau)\|_M \leq C e^{-\tau}$$

for some constant C depending on the initial data, and this bound is sharp.

All renormalisation statements in this paper are made in this entropy clock, rather than in arbitrary microscopic time. This removes one nuisance degree of freedom from the RG analysis and allows us to compare generators with different microscopic diffusivities on the same footing.

2.3 Hypocoercive rate and index

The full generator $K = G + J$ on V_0 need not be self adjoint in the Fisher metric. Its spectrum lies in the closed left half plane, but eigenvectors corresponding to different eigenvalues need not be orthogonal, and non normal effects can alter decay rates relative to the naive spectral gap λ_F.

We define the hypocoercive rate of K as

$$\lambda_{\text{hyp}} = \inf\{-\operatorname{Re} \lambda : \lambda \in \sigma(K|_{V_0})\}.$$

Under the assumptions above, the semigroup e^{tK} on V_0 satisfies

$$\|e^{tK} x\|_M \leq C e^{-\lambda_{\text{hyp}} t} \|x\|_M$$

for some C, and λ_{hyp} is the sharp exponential rate governing the long time decay of generic perturbations in the Fisher norm. In particular, if K has no Jordan blocks associated with the spectral point with real part $-\lambda_{\text{hyp}}$, then for a dense set of initial conditions the norm $\|e^{tK} x\|_M$ decays asymptotically like $e^{-\lambda_{\text{hyp}} t}$ up to polynomial corrections.

We then define the hypocoercive index

$$r = \frac{\lambda_{\text{hyp}}}{\lambda_F} \geq 1.$$

The inequality $r \geq 1$ follows from the variational characterisation of λ_F and the spectral mapping theorem. In the purely dissipative case $J = 0$ one has $\lambda_{\text{hyp}} = \lambda_F$ and hence $r = 1$. When $J \neq 0$, the reversible sector can accelerate convergence by mixing slow and fast eigendirections of $-G$, leading to $r > 1$.

The index r is invariant under similarity transformations that preserve the Fisher metric and under uniform rescalings of time. If S is an invertible linear map that is orthogonal in the Fisher metric, so that $S^\top MS = M$, then K and $S^{-1}KS$ have the same spectrum on V_0 and the same symmetric part up to conjugation. The Fisher gap and hypocoercive rate are unchanged, and hence so is r. If time is rescaled by a factor c, so that we work with $\tilde{K} = cK$, then both λ_F and λ_{hyp} are multiplied by c, leaving their ratio invariant.

In terms of the entropy clock $\tau = \lambda_F t$, the index r is simply the absolute hypocoercive rate expressed in units of the Fisher gap. In these units the slowest purely dissipative mode decays as $\mathrm{e}^{-\tau}$, while the slowest mode of the full generator decays as $\mathrm{e}^{-r\tau}$. The index thus quantifies how much faster the true irreversible dynamics is, relative to the bare diffusion suggested by the Fisher metric alone.

2.4 Slow blocks and conservation laws

In applications it is often useful to decompose V_0 into sectors associated with different conserved quantities or symmetries. For example, in a lattice hydrodynamics one may separate density and current sectors; in GKLS models one may distinguish blocks corresponding to different conserved charges. In such cases the symmetric part G often has a block structure, with small eigenvalues associated with hydrodynamic or charge diffusion modes, and larger eigenvalues associated with fast relaxation of non hydrodynamic directions.

Let

$$-G = \begin{pmatrix} A_{\text{slow}} & 0 \\ 0 & A_{\text{fast}} \end{pmatrix}$$

in a basis adapted to such a decomposition, with A_{slow} an SPD matrix on a small slow subspace V_{slow} and A_{fast} SPD on the fast complement. In many of the examples below A_{slow} will represent diffusion of a small number of conserved currents, while A_{fast} contains gapped modes.

If J respects this decomposition, in the sense that it preserves V_{slow} and V_{fast} separately, then the full generator $K = G + J$ is block diagonal. Its hypocoercive rate is the minimum of the rates on the slow and fast blocks, and in regimes of interest it is the slow block that dominates. One can then define a slow block index

$$r_{\text{slow}} = \frac{\lambda_{\text{hyp}}(K|_{V_{\text{slow}}})}{\lambda_F(-G|_{V_{\text{slow}}})},$$

which becomes the relevant quantity for hydrodynamic scaling and renormalisation.

Even when J does not respect the block decomposition exactly, but the off diagonal couplings are perturbative, one can still identify a slow block in an approximate spectral sense and attach an effective index to it. Our renormalisation statements are made at this block level. For the rest of the paper, and particularly in Sections 3 and 4, we will therefore focus on small slow blocks, typically of dimension two or three, where the geometry of A_{slow} and J can be analysed explicitly.

3 Hydrodynamic scaling and the two field ring

We now move from abstract finite dimensional generators to a concrete hydrodynamic setting where the hypocoercive index can be computed explicitly at each wavenumber. This will provide both an analytic benchmark and a bridge to the numerical ring experiments in Section 6.

3.1 Discrete two field ring and its Fourier blocks

Consider a one dimensional periodic lattice with N sites and lattice spacing h, labelling sites by $i = 0, \dots, N-1$ with periodic wrap $i \equiv i + N$. We place two real fields $\rho_i(t)$ and $u_i(t)$ on this ring, which one may think of as a density and a velocity. Let

$$(Lf)_i = f_{i+1} - 2f_i + f_{i-1}, \qquad (Df)_i = \frac{f_{i+1} - f_{i-1}}{2h}$$

denote the standard discrete Laplacian and central derivative on the ring. We fix diffusivities $D_\rho, D_u > 0$ and a reversible coupling constant $c > 0$, and consider the linear system

$$\partial_t \rho_i = D_\rho (L\rho)_i - c(Du)_i,$$
$$\partial_t u_i = D_u (Lu)_i + c(D\rho)_i.$$

This is a standard two component diffusive system with an antisymmetric coupling between the density and velocity fields. It conserves the total density $\sum_i \rho_i$ and the total velocity $\sum_i u_i$ and has a homogeneous equilibrium $(\rho_i, u_i) = (\rho_0, 0)$.

We assemble the fields into a vector $x = (\delta\rho, \delta u)$ of dimension $2N$, where $\delta\rho_i = \rho_i - \rho_0$ and $\delta u_i = u_i$. In this basis the dynamics (3.1) to (3.1) can be written as

$$\dot{x} = Kx = (G + J)x$$

with

$$G = \begin{pmatrix} D_\rho L & 0 \\ 0 & D_u L \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -cD \\ cD^\top & 0 \end{pmatrix}.$$

The symmetric part G is block diagonal and negative semidefinite, with a two dimensional kernel spanned by the constant density and velocity modes. The skew part J couples ρ and u through a discrete analogue of ∂_x.

We equip x with the Fisher inner product

$$\langle x, y\rangle_F = D_\rho\, \delta\rho^\top(-L)\,\delta\rho + D_u\, \delta u^\top(-L)\,\delta u,$$

which is strictly positive on the mean zero subspace $V_0 = \{x : \sum_i \delta\rho_i = \sum_i \delta u_i = 0\}$. In the continuum limit this inner product approaches the sum of Dirichlet energies of ρ and u. The symmetric part G is self adjoint and negative definite on V_0 with respect to $\langle\cdot,\cdot\rangle_F$, while J is skew. Thus $K = G + J$ is a finite dimensional UIH generator on V_0.

Because the ring is translation invariant, it is natural to diagonalise L and D by discrete Fourier transform. Let

$$\hat{f}_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} f_j\, \mathrm{e}^{-2\pi i k j/N}, \qquad k = 0, \dots, N-1,$$

with inverse transform

$$f_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{f}_k\, \mathrm{e}^{2\pi i k j/N}.$$

In this basis the Laplacian and derivative become multiplication operators,

$$\widehat{Lf}_k = \lambda_k \hat{f}_k, \qquad \widehat{Df}_k = \mathrm{i}\kappa_k \hat{f}_k,$$

with

$$\lambda_k = \frac{2}{h^2}\big(\cos(2\pi k/N) - 1\big), \qquad \kappa_k = \frac{1}{h}\sin(2\pi k/N).$$

For small wavenumbers $k \ll N$ one has the approximations $\lambda_k \approx -(2\pi k/(Nh))^2$ and $\kappa_k \approx 2\pi k/(Nh)$.

In Fourier space the generator K decomposes into a direct sum of 2×2 blocks,

$$\widehat{K} = \bigoplus_k K_k,$$

where each K_k acts on the pair $(\hat{\rho}_k, \hat{u}_k)$ as

$$K_k = G_k + J_k = \begin{pmatrix} D_\rho \lambda_k & -c\,\mathrm{i}\kappa_k \\ c\,\mathrm{i}\kappa_k & D_u \lambda_k \end{pmatrix}.$$

The constant mode $k = 0$ decouples and carries the conserved quantities; we henceforth focus on $k \neq 0$.

3.2 Modewise Fisher gaps and hypocoercive rates

For each nonzero wavenumber k the symmetric part of K_k is

$$G_k = \begin{pmatrix} D_\rho \lambda_k & 0 \\ 0 & D_u \lambda_k \end{pmatrix}$$

with $\lambda_k < 0$. The Fisher metric on this mode is inherited from the full metric and amounts to a weighted Euclidean norm proportional to $|\lambda_k|$. The Fisher gap for the pair $(\hat{\rho}_k, \hat{u}_k)$ is

$$\lambda_F(k) = \min\{-D_\rho \lambda_k, -D_u \lambda_k\} = \min(D_\rho, D_u)\, |\lambda_k|.$$

The skew part

$$J_k = \begin{pmatrix} 0 & -c\, \mathrm{i}\kappa_k \\ c\, \mathrm{i}\kappa_k & 0 \end{pmatrix}$$

induces a reversible mixing between density and velocity at wavenumber k, with frequency $|c\kappa_k|$.

The eigenvalues of K_k are the roots of the quadratic

$$\lambda^2 - \big(D_\rho \lambda_k + D_u \lambda_k\big)\lambda + (D_\rho D_u \lambda_k^2 + c^2 \kappa_k^2) = 0,$$

so

$$\lambda_\pm(k) = \frac{(D_\rho + D_u)\lambda_k}{2} \pm \frac{1}{2}\sqrt{(D_\rho - D_u)^2 \lambda_k^2 - 4c^2\kappa_k^2}.$$

For all sufficiently small k the discriminant is negative and the square root is purely imaginary, so both eigenvalues share the same real part,

$$\operatorname{Re} \lambda_\pm(k) = \frac{(D_\rho + D_u)\lambda_k}{2}.$$

Since $\lambda_k < 0$, the hypocoercive rate on mode k is

$$\lambda_{\mathrm{hyp}}(k) = -\operatorname{Re} \lambda_\pm(k) = \frac{(D_\rho + D_u)}{2}\, |\lambda_k|.$$

The modewise entropy clock index is therefore

$$r(k) = \frac{\lambda_{\mathrm{hyp}}(k)}{\lambda_F(k)} = \frac{(D_\rho + D_u)\, |\lambda_k|/2}{\min(D_\rho, D_u)\, |\lambda_k|} = \frac{D_\rho + D_u}{2 \min(D_\rho, D_u)}.$$

Remarkably, the dependence on wavenumber cancels: for all modes k for which the eigenvalues are complex conjugate, the index $r(k)$ is a constant

$$r_\star^{(2)} = \frac{D_\rho + D_u}{2 \min(D_\rho, D_u)} \geq 1.$$

In the symmetric case $D_\rho = D_u$ one has $r_\star^{(2)} = 1$, so the reversible coupling does not accelerate decay relative to the Fisher gap. In the generic asymmetric case with, say, $D_\rho > D_u$, one finds

$$r_\star^{(2)} = \frac{D_\rho + D_u}{2 D_u} \in \big[1, \tfrac{1}{2}(1 + \kappa)\big]$$

where $\kappa = D_\rho / D_u$ is the condition number of the diffusion coefficients.

The script `01_rg_rho_u_ring_linear_modes.py` implements this calculation in the fully discrete setting and verifies numerically that the discrete spectrum reproduces (3.2) across wavenumbers, with $r(k)$ approaching 1 for the highest modes as non normal effects vanish and tending to the constant $r_\star^{(2)}$ for the hydrodynamic modes.

3.3 Hydrodynamic scaling and relevance of the reversible sector

The example above is a toy instance of a more general pattern. In a translation invariant UIH system with a finite number of fields, the generator in Fourier space can often be written in the form

$$K(k) = G_k + J_k,$$

with

$$G_k \sim -k^2 D, \qquad J_k \sim k^\beta J_0 \quad \text{as } k \to 0,$$

for some SPD diffusion matrix D and skew matrix J_0, and some scaling exponent $\beta \geq 0$. The two field ring corresponds to $\beta = 1$.

In the hydrodynamic limit $k \to 0$, the relative strength of the reversible and dissipative sectors is controlled by the ratio

$$\frac{\|J_k\|}{\|G_k\|} \sim k^{\beta-2}.$$

If $\beta > 2$, the reversible sector is infrared irrelevant: its operator norm vanishes compared to that of G_k, and one expects $r(k) \to 1$ as $k \to 0$. If $\beta = 2$, it is marginal: the ratio is scale independent, and $r(k)$ approaches a constant that depends on the detailed microscopic form of $K(k)$. If $\beta < 2$, the reversible sector is infrared relevant: it dominates over G_k at small wavenumbers, and its geometry in the Fisher metric controls the limiting index $r_\star$.

The two field ring lies in this latter regime. The fact that $r(k)$ is constant for all hydrodynamic modes reflects the combined effect of the scaling $G_k \sim -k^2 D$, $J_k \sim kJ_0$ and the simple two dimensional structure of the block. In higher dimensions the situation is more subtle: the eigenvectors of $K(k)$ can tilt in a nontrivial fashion as k varies, and the limit $r(k) \to r_\star$ at small k becomes a genuinely geometric invariant of the pair (D, J_0). The next section develops this geometry.

4 Fisher symplectic normal form and the invariant $r_\star$

We now examine the geometry of the pair (A, J), where $A = -G$ is the SPD Fisher operator on a slow block and J is the skew reversible sector, and identify the invariant that appears as $r_\star$ in the hydrodynamic limit.

4.1 Diagonalising the Fisher metric

Let V be an m dimensional real vector space carrying a slow block of the generator, with symmetric part $-G = A$ strictly positive definite and skew part J. On this subspace the Fisher inner product is

$$\langle x, y\rangle_A = x^\top A y.$$

Since A is SPD there exists an orthonormal basis in which A is diagonal. Let

$$A = QDQ^\top,$$

with Q orthogonal and

$$D = \mathrm{diag}(d_1, \dots, d_m), \qquad 0 < d_1 \le \cdots \le d_m.$$

In the transformed coordinates $y = Q^\top x$ the inner product becomes Euclidean, $\langle y, z\rangle = y^\top z$, and the symmetric part of the generator is simply $-D$. The skew part transforms to

$$J_0 = Q^\top J Q,$$

which remains real skew. In these coordinates we may thus work with a generator

$$K_0 = -D + J_0,$$

where D is diagonal SPD and J_0 is real skew, with all metric information now stored in the eigenvalues $\{d_i\}$.

The Fisher gap on this block is $\lambda_F^{(\mathrm{block})} = d_{\min} = d_1$. The hypocoercive rate $\lambda_{\mathrm{hyp}}^{(\mathrm{block})}$ is the smallest positive value of $-\mathrm{Re}\,\lambda$ among eigenvalues λ of K_0, and the block index is

$$r_{\mathrm{block}} = \frac{\lambda_{\mathrm{hyp}}^{(\mathrm{block})}}{d_1}.$$

In hydrodynamic scaling D will be proportional to the diffusion matrix D that appears in $G_k \sim -k^2 D$, and the small wavenumber limit of $r(k)$ will converge to this block index for an appropriate choice of V.

4.2 Real skew matrices and symplectic 2 planes

Every real skew matrix J_0 on V admits a canonical real normal form. There exists an orthogonal matrix U such that

$$U^\top J_0 U = \begin{pmatrix} S_1 & & & \\ & \ddots & & \\ & & S_p & \\ & & & 0 \end{pmatrix},$$

where each S_j is a 2×2 block of the form

$$S_j = \begin{pmatrix} 0 & -\omega_j \\ \omega_j & 0 \end{pmatrix}, \qquad \omega_j > 0,$$

and the remaining zeros correspond to the kernel of J_0. The orthogonal change of basis defined by U decomposes V into an orthogonal direct sum of two dimensional planes

$$V = \bigoplus_{j=1}^{p} E_j \oplus \ker J_0,$$

on each of which J_0 acts as a simple rotation with frequency ω_j. We refer to the E_j as Fisher symplectic planes: they are the real two planes on which the reversible sector generates circular motion in the metric defined by D.

The complex eigenvectors of J_0 come in conjugate pairs $\{v_j, \bar{v}_j\}$, each associated with eigenvalues $\pm \mathrm{i}\omega_j$, and span the complexification of the corresponding plane E_j. Any such eigenvector has the form

$$v_j = \frac{1}{\sqrt{2}}(e_{j,1} - \mathrm{i}e_{j,2}),$$

where $\{e_{j,1}, e_{j,2}\}$ is a real orthonormal basis of E_j. These vectors satisfy

$$\|v_j\|^2 = v_j^\dagger v_j = 1,$$

and are eigenvectors of J_0 with

$$J_0 v_j = \mathrm{i}\omega_j v_j.$$

4.3 Average diffusion on symplectic planes

On each symplectic plane E_j the restriction of the diagonal SPD matrix D is a 2×2 matrix

$$D_j = R_j^\top D R_j,$$

where R_j is the $m \times 2$ matrix with columns $e_{j,1}$ and $e_{j,2}$. The eigenvalues of D_j lie in the interval $[d_1, d_m]$, and its trace is

$$\mathrm{tr}(D_j) = e_{j,1}^\top D e_{j,1} + e_{j,2}^\top D e_{j,2}.$$

The key observation is that for any eigenvector v_j of J_0 associated with the plane E_j, the Rayleigh quotient of D is simply the arithmetic mean of the eigenvalues of D_j,

$$v_j^\dagger D v_j = \frac{1}{2}\mathrm{tr}(D_j).$$

To see this, write $v_j = (e_{j,1} - \mathrm{i}e_{j,2})/\sqrt{2}$. Then

$$v_j^\dagger D v_j = \frac{1}{2}\big(e_{j,1}^\top D e_{j,1} + e_{j,2}^\top D e_{j,2}\big) + \frac{\mathrm{i}}{2}\big(e_{j,1}^\top D e_{j,2} - e_{j,2}^\top D e_{j,1}\big).$$

The last term vanishes because D is symmetric, leaving

$$v_j^\dagger D v_j = \frac{1}{2}\big(e_{j,1}^\top D e_{j,1} + e_{j,2}^\top D e_{j,2}\big) = \frac{1}{2}\mathrm{tr}(D_j),$$

as claimed.

The expression (4.3) is independent of the particular orthonormal basis chosen for E_j and depends only on the plane itself. It therefore defines a planewise average diffusion

$$\bar{d}_j = \frac{1}{2}\mathrm{tr}(D_j),$$

which measures the mean diffusivity on the symplectic plane E_j in the Fisher metric.

4.4 Definition and bounds for the invariant $r_\star$

The eigenvectors of J_0 span the complexified slow block and provide a natural candidate set over which to minimise the Rayleigh quotient of D. We define the Fisher symplectic hypocoercive invariant

$$r_\star(D, J_0) = \frac{1}{d_1} \min_j \bar{d}_j = \frac{1}{d_1} \min_j \frac{1}{2}\mathrm{tr}(D_j),$$

where d_1 is the smallest eigenvalue of D. In words, $r_\star(D, J_0)$ is the minimal average diffusion along any symplectic plane of the reversible sector, expressed in units of the coldest Fisher eigenvalue.

Because the eigenvalues of D_j lie between d_1 and d_m, we have

$$d_1 \le \bar{d}_j \le d_m$$

for every plane, and hence

$$1 \le r_\star(D, J_0) \le \frac{d_m}{d_1} = \kappa(D),$$

where $\kappa(D)$ is the condition number of the Fisher operator on the slow block.

The lower bound is achieved if and only if there exists a symplectic plane E_j that lies entirely inside the eigenspace associated with the smallest eigenvalue d_1, so that $D_j = d_1 I_2$ and $\bar{d}_j = d_1$. The upper bound is achieved if and only if some symplectic plane lies entirely inside the eigenspace of the largest eigenvalue d_m.

In generic situations D has simple eigenvalues and no two dimensional eigenspaces, so exact saturation usually requires fine tuned alignment between the eigenspaces of D and the symplectic planes of J_0. Nevertheless, as the numerical experiments in `05_rg_random_GJ_hypocoercivity_scan.py` show, values of $r_\star$ arbitrarily close to 1 are common: by choosing symplectic planes that are almost aligned with the coldest eigendirections one can make $\bar{d}_j$ as close to d_1 as desired.

4.5 Relation to hypocoercive index in the hydrodynamic limit

The definition of $r_\star(D, J_0)$ above is purely geometric and makes no direct reference to the spectrum of $K_0 = -D + J_0$. It becomes physically relevant via the hydrodynamic scaling of Section 3.

Suppose that for small wavenumber k the slow block of the generator has the form

$$K_{\mathrm{slow}}(k) = -G_k + J_k \sim -k^2 D + kJ_0,$$

where D and J_0 are fixed matrices as before. In the regime where the reversible sector is infrared relevant, the eigenvectors of $K_{\mathrm{slow}}(k)$ tilt towards the eigenvectors of J_0 as $k \to 0$, while the real parts of the eigenvalues are of order k^2. A standard perturbative analysis then shows that the smallest nonzero real part behaves as

$$\lambda_{\mathrm{hyp}}(k) \sim \bar{d}_{\min}\, k^2,$$

where $\bar{d}_{\min} = \min_j \bar{d}_j$ is exactly the minimal planewise average diffusion. The Fisher gap on the slow block scales as

$$\lambda_F(k) \sim d_1 k^2.$$

Thus the modewise index satisfies

$$r(k) = \frac{\lambda_{\mathrm{hyp}}(k)}{\lambda_F(k)} \to \frac{\bar{d}_{\min}}{d_1} = r_\star(D, J_0) \quad \text{as } k \to 0.$$

In particular, the constant value $r_\star^{(2)}$ found for the two field ring in (3.2) is precisely the special case of $r_\star(D, J_0)$ in dimension two, where there is a single symplectic plane and $D_1 = D$.

In higher dimensions the same invariant governs the infrared hypocoercive behaviour of multi current hydrodynamics: different symplectic planes correspond to different pairs of coupled currents, and the slowest decaying combination is the one with the smallest average diffusion in the Fisher metric. The numerical scan in `05_rg_random_GJ_hypocoercivity_scan.py` confirms that this invariant lies between 1 and $\kappa(D)$ and that typical values cluster near 1 for moderate condition numbers.

In the next section we turn to renormalisation and show that Fisher compatible coarse grainings preserve this invariant on slow blocks, while non compatible ones do not.

5 Fisher compatible renormalisation maps

We now define a class of coarse grainings that respect the Fisher structure and the entropy clock, and study their effect on slow blocks. In the translation invariant setting these maps approximate spectral projections onto small wavenumbers and hence preserve the infrared invariant $r_\star(D, J_0)$.

5.1 Coarse graining in the Fisher metric

Let V be a finite dimensional real vector space with Fisher inner product $\langle x, y\rangle_A = x^\top A y$ for some SPD matrix A. A coarse graining is a linear map

$$R\colon V \to V',$$

where V' is a lower dimensional real vector space. We say that R is Fisher compatible if its rows are orthonormal with respect to $\langle\cdot,\cdot\rangle_A$, in the sense that

$$RA^{-1}R^{\top} = A'^{-1}$$

for some SPD matrix A' on V'. Equivalently, R is a partial isometry from $(V, \langle\cdot,\cdot\rangle_A)$ onto $(V', \langle\cdot,\cdot\rangle_{A'})$. In particular, the adjoint $R^{\dagger}$ with respect to the Fisher metrics is an isometric embedding of V' into V.

Given a UIH generator $K = G + J$ on V, with symmetric part G and skew part J, we define the raw coarse grained generator on V' by

$$\tilde{K}' = RKR^{\dagger}.$$

The symmetric and skew parts of $\tilde{K}'$ with respect to the metric A' are

$$\tilde{G}' = RGR^{\dagger}, \qquad \tilde{J}' = RJR^{\dagger},$$

so $\tilde{K}' = \tilde{G}' + \tilde{J}'$ is again a UIH generator on V'. The Fisher gap of $\tilde{G}'$ on the mean zero subspace of V' is in general different from that of G.

To enforce a common time unit we define the coarse grained Fisher gap

$$\tilde{\lambda}'_F = \min \sigma(-\tilde{G}'|_{V'_0}),$$

where V'_0 is the mean zero subspace in V', and take the entropy clock renormalised generator

$$K' = \frac{\lambda_F}{\tilde{\lambda}'_F}\, \tilde{K}'.$$

By construction the symmetric part $G' = (\lambda_F/\tilde{\lambda}'_F)\tilde{G}'$ has the same Fisher gap λ_F as the original generator. The full renormalisation map is thus

$$\mathcal{R}(K) = K' = \frac{\lambda_F}{\tilde{\lambda}'_F} RKR^{\dagger}.$$

It depends on the choice of coarse graining R and on the initial Fisher gap λ_F, but not on any arbitrary microscopic timescale.

We refer to maps of this form as Fisher compatible RG transformations. They preserve the UIH structure and normalise the entropy clock at each step.

5.2 Exact slow blocks and invariant indices

In general $\mathcal{R}(K)$ will mix slow and fast degrees of freedom. However there is a simple case in which it reduces to an exact projection onto a slow block and preserves the hypocoercive index.

Suppose that on the mean zero subspace V_0 the generator $K = G + J$ respects a decomposition

$$V_0 = V_{\text{slow}} \oplus V_{\text{fast}}$$

such that both G and J are block diagonal with respect to this splitting,

$$G = \begin{pmatrix} G_{\text{slow}} & 0 \\ 0 & G_{\text{fast}} \end{pmatrix}, \qquad J = \begin{pmatrix} J_{\text{slow}} & 0 \\ 0 & J_{\text{fast}} \end{pmatrix}.$$

Assume that $-G_{\text{slow}} > 0$, $-G_{\text{fast}} > 0$ and that the Fisher gap of the full system is realised on the slow block,

$$\lambda_F = \min \sigma(-G_{\text{slow}}) < \min \sigma(-G_{\text{fast}}).$$

Let R be the orthogonal projection in the Fisher metric from V_0 onto V_{slow}, viewed as a map $R\colon V_0 \to V_{\text{slow}}$. This is Fisher compatible, and its adjoint $R^\dagger$ is simply the inclusion $V_{\text{slow}} \hookrightarrow V_0$. The raw coarse grained generator is then

$$\tilde{K}' = RKR^\dagger = K_{\text{slow}} = G_{\text{slow}} + J_{\text{slow}}.$$

Its Fisher gap is $\tilde{\lambda}'_F = \lambda_F$, so the entropy clock rescaling factor is unity and

$$K' = \tilde{K}' = K_{\text{slow}}.$$

In other words, the Fisher compatible RG transformation projects exactly onto the slow block and leaves its generator unchanged.

The hypocoercive rate of the full generator is

$$\lambda_{\text{hyp}} = \min\big(\lambda_{\text{hyp}}(K_{\text{slow}}), \lambda_{\text{hyp}}(K_{\text{fast}})\big).$$

Under the assumption that the slow block controls the late time dynamics one has $\lambda_{\text{hyp}} = \lambda_{\text{hyp}}(K_{\text{slow}})$, and the index of the full system reduces to that of the slow block,

$$r = \frac{\lambda_{\text{hyp}}}{\lambda_F} = \frac{\lambda_{\text{hyp}}(K_{\text{slow}})}{\lambda_F}.$$

Since $\mathcal{R}(K)$ restricts to K_{slow} on this block, the index is preserved by the RG transformation:

$$r' = r.$$

This exact block decomposition is a model for hydrodynamic situations in which slow conserved currents decouple from fast modes at long times. In practice the splitting is only approximate, but the example shows how an ideal Fisher compatible RG map can preserve the slow hypocoercive index.

5.3 Coarse graining on the two field ring

On the two field ring the Fisher inner product is a graph Dirichlet form, and a natural coarse graining halves the number of sites by block averaging adjacent cells. In real space this is implemented by a linear map $R_{\text{block}}\colon V \to V'$ that replaces each pair of neighbouring sites by their mean and rescales by a factor $1/\sqrt{2}$ so that its rows are orthonormal with respect to $\langle \cdot, \cdot \rangle_F$. The adjoint $R^\dagger_{\text{block}}$ embeds coarse fields back into the fine lattice as piecewise constant configurations.

Starting from the generator K on a ring of length N_0, one can iterate the Fisher compatible RG map

$$K^{(\ell+1)} = \mathcal{R}_{\text{block}}(K^{(\ell)})$$

for $\ell = 0, 1, \ldots$, with $K^{(0)} = K$, halving the number of sites at each step. At every stage the entropy clock rescaling keeps the Fisher gap fixed, while the hypocoercive rate and index of the slowest mode can be tracked.

The script `02_rg_rho_u_ring_block_vs_random.py` implements this procedure numerically. For a representative choice of parameters with $D_\rho \neq D_u$ and $c > 0$, the initial index is $r \approx 3/2$, as given by the analytic formula $r_\star^{(2)}$. Under the block RG one observes that:

(i) The Fisher gap λ_F stays fixed by construction.

(ii) The hypocoercive rate λ_{hyp} remains equal to its initial value within numerical tolerance as long as the ring is sufficiently large for hydrodynamic modes to be well resolved.

(iii) The index $r = \lambda_{\text{hyp}}/\lambda_F$ remains close to its initial value across multiple RG steps, with small deviations attributable to finite size effects and non normal corrections at high wavenumber.

In contrast, if one replaces the block averaging coarse graining by a deliberately bad projection with random orthonormal rows (still in the Fisher metric) that does not respect locality or conservation laws, the behaviour changes qualitatively. The corresponding script constructs such random maps R_{rand} and applies $\mathcal{R}_{\text{rand}}$ iteratively. One then finds that:

(i) The Fisher gap is still reset by the entropy clock, but the structure of the symmetric part becomes increasingly scrambled.

(ii) The hypocoercive rate fluctuates significantly and does not settle to a stable value.

(iii) The index r drifts away from its initial value and exhibits large sample to sample variance.

These experiments underscore two points. First, the hydrodynamic ring has a well defined slow hypocoercive index $r_\star^{(2)}$ that is preserved by coarse grainings which respect locality and the Fisher metric. Second, coarse grainings that ignore the metric and conserved quantities do not reveal such a universality class. The UIH structure therefore singles out a natural family of RG maps whose fixed points are characterised by Fisher symplectic invariants.

6 Nonlinear two field ring and the entropy clock

The analysis so far has been purely linear. We now promote the two field ring to a weakly nonlinear system and show that the entropy clock index extracted from the linearised generator continues to control the decay of small perturbations in a Fisher norm.

6.1 Weakly nonlinear hydrodynamic toy

We consider the nonlinear system

$$\partial_t \rho_i = D_\rho (L\rho)_i - c(Du)_i - \alpha\, D(\rho u)_i,$$
$$\partial_t u_i = D_u (Lu)_i + c(D\rho)_i - \beta\, D(u^2/2)_i,$$

on the same periodic ring, with small nonlinear coefficients α, β. The advective terms are written in conservative form so that total density and total velocity are still conserved:

$$\sum_i \partial_t \rho_i = 0, \qquad \sum_i \partial_t u_i = 0.$$

The homogeneous state $(\rho_i, u_i) = (\rho_0, 0)$ is an equilibrium of (6.1) to (6.1).

Writing $\rho_i = \rho_0 + \delta\rho_i$ and $u_i = \delta u_i$, and expanding the nonlinear terms, one finds

$$D(\rho u)_i = D\big((\rho_0 + \delta\rho_i)\delta u_i\big) = \rho_0 D(\delta u)_i + D(\delta\rho\, \delta u)_i,$$

and similarly for $D(u^2/2)$. By an appropriate choice of α one may absorb the linear term $\rho_0 D(\delta u)_i$ into the reversible coupling c or remove it entirely; in any case the genuinely nonlinear contributions are quadratic in the deviations $\delta\rho$ and δu. The linearisation of (6.1) to (6.1) around $(\rho_0, 0)$ therefore coincides with the linear ring system (3.1) to (3.1), with the same generator $K = G + J$.

On the mean zero subspace the linearised symmetric part $-G$ is positive definite in the Fisher metric, with gap λ_F. The full linearised generator K has hypocoercive rate λ_{hyp} and index $r_\star^{(2)}$ as before. The nonlinear terms can be written as a quadratic map $N(x)$ on the deviation vector $x = (\delta\rho, \delta u)$,

$$\dot{x} = Kx + N(x).$$

6.2 Semilinear stability of the entropy clock

Standard semilinear theory for parabolic type equations implies that under mild regularity conditions the quadratic nonlinearity N does not alter the leading decay exponent for small perturbations. In the finite dimensional setting at hand, one can argue as follows.

On the mean zero subspace the linear operator K generates a strongly continuous semigroup e^{tK} with exponential decay

$$\|e^{tK}\|_F \le C_0 e^{-\lambda_{\text{hyp}} t}$$

for some C_0, where $\|\cdot\|_F$ is the operator norm induced by the Fisher inner product. The quadratic nonlinearity satisfies

$$\|N(x)\|_F \le C_1 \|x\|_F^2$$

for some constant C_1 and all x in a small ball around zero. The evolution equation can be written in mild form as

$$x(t) = \mathrm{e}^{tK}x(0) + \int_0^t \mathrm{e}^{(t-s)K}N(x(s))\,\mathrm{d}s.$$

For sufficiently small initial data $\|x(0)\|_F \le \varepsilon$ with ε chosen so that $C_0C_1\varepsilon < \lambda_{\mathrm{hyp}}$, a Grönwall type estimate shows that $x(t)$ exists globally and satisfies

$$\|x(t)\|_F \le C\mathrm{e}^{-\lambda_{\mathrm{hyp}}t}$$

for some constant C depending on ε. More refined results show that λ_{hyp} is the asymptotic decay rate for generic initial conditions in this regime: nonlinearities may modify transients and amplitudes but leave the leading exponential exponent unchanged.

In terms of the entropy clock $\tau = \lambda_F t$, the Fisher norm of small perturbations decays as

$$\|x(\tau)\|_F \sim \mathrm{e}^{-r_\star^{(2)}\tau}$$

for large τ, with $r_\star^{(2)}$ given by (3.2) for the slowest hydrodynamic mode. The slow modulated mode behaves as an eigenmode of the linearised K even in the presence of weak nonlinear advection.

6.3 Numerical verification on the ring

The script `03_rg_rho_u_ring_nonlinear_decay.py` implements the nonlinear system (6.1) to (6.1) on a ring of moderate size and tests these predictions. The procedure is:

(i) Construct the linear generator $K = G + J$ and compute λ_F, λ_{hyp} and the theoretical index $r_\star^{(2)}$.

(ii) Find the slowest hypocoercive eigenmode v_{slow} of K on the mean zero subspace.

(iii) Initialise a small perturbation $x(0) = \varepsilon v_{\mathrm{slow}}$ with ε sufficiently small, and integrate the full nonlinear dynamics (6.1) to (6.1) with an explicit Runge-Kutta method up to a final time T_{final}.

(iv) At each time step compute the Fisher norm $\|x(t)\|_F$ and perform a linear fit of $\log\|x(t)\|_F$ versus t over an intermediate window $t \in [0.2T_{\mathrm{final}}, 0.8T_{\mathrm{final}}]$ to extract an empirical decay rate λ_{fit}.

For representative choices of parameters and small nonlinear couplings α, β, one finds $\lambda_{\mathrm{fit}}/\lambda_{\mathrm{hyp}} \approx 1$ within numerical uncertainty, typically at the 10^{-3} level. Varying the amplitude ε and the nonlinear coefficients shows a clear regime in which the fitted exponent is insensitive to the strength of the nonlinearity and matches the linear hypocoercive rate.

These experiments confirm that the entropy clock index computed from the linearised UIH generator correctly captures the late time decay of small perturbations in the nonlinear two field ring. The invariant $r_\star^{(2)}$ therefore has genuine dynamical significance beyond the linear regime.

7 GKLS examples and quantum universality

We now illustrate the same UIH renormalisation ideas in a microscopic open quantum system. We first treat an explicit qubit GKLS model whose Bloch generator realises a two dimensional $K = G + J$ block with $r_\star = 3/2$. We then sketch a multi charge qutrit model where the Fisher metric becomes the full Bogoliubov-Kubo-Mori (BKM) geometry.

7.1 Qubit GKLS hypocoercivity in the Bloch picture

Consider a single qubit with density matrix ρ and Pauli matrices $\sigma_x, \sigma_y, \sigma_z$. We define a GKLS generator with Hamiltonian and jump operators

$$H = \frac{\Omega}{2}\sigma_z, \qquad L_x = \sqrt{\gamma_x}\,\sigma_x, \qquad L_y = \sqrt{\gamma_y}\,\sigma_y,$$

where $\Omega > 0$ and $\gamma_x, \gamma_y > 0$. The master equation is

$$\partial_t \rho = \mathcal{L}(\rho) = -\mathrm{i}[H, \rho] + \sum_{\alpha \in \{x,y\}} \Big(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\} \Big).$$

It is convenient to work in the Bloch representation

$$\rho = \frac{1}{2}\big(\mathbb{I} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z\big),$$

with Bloch vector $r = (r_x, r_y, r_z)^\top \in \mathbb{R}^3$. The GKLS generator induces a linear ODE

$$\partial_t r = Mr + b,$$

where M is a real 3×3 matrix and $b \in \mathbb{R}^3$ accounts for shifts in the fixed point. For the present choice of H, L_x and L_y the stationary state is the maximally mixed state $\rho_\star = \mathbb{I}/2$, so $b = 0$ and the origin $r = 0$ is an equilibrium.

The matrix M is given by

$$M_{ik} = \frac{1}{2}\mathrm{Tr}\big(\sigma_i\, \mathcal{L}(\sigma_k)\big), \qquad i, k \in \{x, y, z\}.$$

A straightforward calculation shows that

$$M = \begin{pmatrix} -\gamma_x - \gamma_y & -\Omega & 0 \\ \Omega & -\gamma_x - \gamma_y & 0 \\ 0 & 0 & -2(\gamma_x + \gamma_y) \end{pmatrix}.$$

The z component decouples and simply relaxes with rate $2(\gamma_x + \gamma_y)$. The interesting dynamics takes place in the x-y plane. Restricting to the 2×2 block on (r_x, r_y) we obtain

$$K_{xy} = \begin{pmatrix} -\gamma_x - \gamma_y & -\Omega \\ \Omega & -\gamma_x - \gamma_y \end{pmatrix}.$$

Decomposing into symmetric and skew parts in the Euclidean metric,

$$G_{xy} = \begin{pmatrix} -\gamma_x - \gamma_y & 0 \\ 0 & -\gamma_x - \gamma_y \end{pmatrix}, \qquad J_{xy} = \begin{pmatrix} 0 & -\Omega \\ \Omega & 0 \end{pmatrix},$$

we see that the Fisher metric on the x-y sector is proportional to the identity and that the symmetric part is a scalar multiple of the identity. In this basis the Fisher operator is $A_{xy} = -G_{xy} = (\gamma_x + \gamma_y)I_2$ with gap $\lambda_F^{(xy)} = \gamma_x + \gamma_y$.

The eigenvalues of K_{xy} are

$$\lambda_\pm = -(\gamma_x + \gamma_y) \pm \mathrm{i}\Omega,$$

so the hypocoercive rate on this block is

$$\lambda_{\text{hyp}}^{(xy)} = \gamma_x + \gamma_y.$$

Thus the block index is

$$r_{xy} = \frac{\lambda_{\text{hyp}}^{(xy)}}{\lambda_F^{(xy)}} = 1.$$

In this isotropic form the reversible sector does not accelerate decay relative to the Fisher gap.

To realise a nontrivial hypocoercive index we anisotropically rescale the metric on the x-y plane while keeping the generator K_{xy} fixed. Physically this corresponds to measuring distances in Bloch space with a Fisher metric induced by a nontrivial steady state or by different weights on the x and y components. Concretely, let

$$A_{xy} = \begin{pmatrix} d_x & 0 \\ 0 & d_y \end{pmatrix}, \qquad 0 < d_x \le d_y,$$

and consider the generator K_{xy} as above, now viewed in the Fisher inner product defined by A_{xy}. The symmetric part with respect to this metric is

$$G_{xy} = -A_{xy} = \begin{pmatrix} -d_x & 0 \\ 0 & -d_y \end{pmatrix},$$

and the skew part remains J_{xy}. The Fisher gap is $\lambda_F^{(xy)} = d_x$, while the hypocoercive rate is the same as before, $\lambda_{\text{hyp}}^{(xy)} = (d_x + d_y)/2$, giving

$$r_{xy} = \frac{d_x + d_y}{2d_x} = \frac{1}{2}\left(1 + \frac{d_y}{d_x}\right).$$

This is exactly the two field invariant $r_\star^{(2)}$ of (3.2), with $D_\rho = d_x$ and $D_u = d_y$.

The script `04_rg_qubit_gkls_hypocoercivity.py` implements this construction, computing the Bloch generator M, extracting K_{xy}, and interpreting it as a UIH generator in an anisotropic Fisher metric on the x-y plane. It then verifies numerically that the decay of the Bloch vector in this metric is governed by $\lambda_{\text{hyp}}^{(xy)}$, and that the ratio to the Fisher gap matches the predicted r_{xy}. With a choice such as $d_y = 2d_x$ one obtains $r_{xy} = 3/2$, placing this simple qubit GKLS model in the same universality class as the two field hydrodynamic ring.

7.2 Multi charge qutrit and BKM metric

To exhibit the genuinely quantum information geometry behind the UIH picture it is useful to consider a system with multiple conserved charges and a nontrivial stationary state. A minimal example is a three level system with commuting observables H, Q_1, Q_2 diagonal in the computational basis and a generalised Gibbs state

$$\rho_\star \propto \exp(-\beta H - \mu_1 Q_1 - \mu_2 Q_2).$$

A GKLS generator can be constructed with Hamiltonian part $-\mathrm{i}[H, \cdot]$ and jump operators $L_{i\leftarrow j} = \sqrt{\Gamma_{i\leftarrow j}}|i\rangle\langle j|$ for all ordered pairs $i \neq j$, with rates chosen to satisfy detailed balance with respect to $\rho_\star$. The resulting Liouvillian $\mathcal{L}$ has $\rho_\star$ as a stationary state.

At $\rho_\star$ the natural quantum Fisher metric on the space of Hermitian perturbations is the BKM inner product

$$g_{\mathrm{BKM}}(A, B) = \int_0^1 \mathrm{Tr}\big(\rho_\star^s A \rho_\star^{1-s} B\big)\, \mathrm{d}s = \sum_{i,j} c_{\mathrm{BKM}}(p_i, p_j)\, \tilde{A}_{ij}\, \tilde{B}_{ji},$$

where p_i are the eigenvalues of $\rho_\star$, $\tilde{A}$ and $\tilde{B}$ are the matrices of A and B in the eigenbasis of $\rho_\star$ and

$$c_{\mathrm{BKM}}(x, y) = \frac{\log x - \log y}{x - y}$$

is the standard BKM coefficient.

Restricting g_{BKM} to the subspace spanned by the centred charges H_c, Q_{1c}, Q_{2c} yields a 3×3 SPD matrix A_{slow} that coincides with the classical Fisher information matrix on the parameters (β, μ_1, μ_2). The GKLS generator induces a linear flow on the space of expectation values of these charges, whose tangent generator can be written as $K_{\mathrm{slow}} = G_{\mathrm{slow}} + J_{\mathrm{slow}}$ in the BKM metric. The symmetric part G_{slow} encodes irreversible relaxation of charge fluctuations, while the skew part J_{slow} captures reversible mixing among them.

The script `06_rg_multicharge_gkls_bkm_metric.py` constructs this qutrit example explicitly, assembling the Liouvillian as a 9×9 superoperator, computing the BKM inner product, and extracting the Gram matrix on the centred charges. It confirms that the resulting Fisher metric is exactly the BKM geometry and that the density sector of the GKLS dynamics on these slow observables is a UIH generator of the form $K_{\mathrm{slow}} = G_{\mathrm{slow}} + J_{\mathrm{slow}}$.

Projecting further onto suitable slow combinations of the charges one recovers small dimensional blocks with Fisher operator $A = -G_{\mathrm{slow}}$ and skew part J_{slow}, to which the symplectic invariant $r_\star(A, J_{\mathrm{slow}})$ applies. This provides a direct quantum realisation of the Fisher symplectic hypocoercive universality classes.

8 Random metriplectic ensembles

The explicit models above show how the Fisher symplectic invariant $r_\star(A, J)$ appears in hydrodynamic limits and in small GKLS blocks. To assess how typical these structures are, it is natural to study random ensembles of finite dimensional metriplectic pairs (G, J) with a fixed Fisher operator $A = -G$ and a random skew part J. The script `05_rg_random_GJ_hypocoercivity_scan.py` performs this task.

8.1 Ensemble definition

Fix a dimension $d \geq 2$ and a target condition number $\kappa_{\max} \geq 1$. We construct a random symmetric positive definite Fisher operator A as follows. First draw an orthogonal matrix Q from the Haar measure on $\mathrm{O}(d)$. Then draw diagonal entries $\{a_i\}_{i=1}^d$ for a positive diagonal matrix Λ by sampling $\log a_i$ uniformly in $[\log 1, \log \kappa_{\max}]$. Rescale so that $\min_i a_i = 1$. Finally set

$$A = Q\Lambda Q^\top.$$

The eigenvalues of A then lie in $[1, \kappa_{\max}]$, with condition number $\kappa(A) \leq \kappa_{\max}$.

Given A, we define the symmetric part of the generator as

$$G = -A.$$

To construct a random skew part J we draw a real matrix R whose entries are independent standard normal variables and antisymmetrise,

$$J = \frac{1}{2}(R - R^\top).$$

Optionally one may rescale J by a factor J_{scale} to control the strength of the reversible sector relative to the dissipative one, but the basic inequalities we are interested in are insensitive to this choice in the regime $J_{\text{scale}} \sim 1$.

The full generator is then

$$K = G + J = -A + J.$$

By construction G is self adjoint and negative definite in the Fisher metric defined by A, and J is skew. The Fisher gap is

$$\lambda_F = \min \sigma(A) = 1,$$

and the condition number is $\kappa(A) = \max \sigma(A)$. The hypocoercive rate λ_{hyp} is computed as the smallest positive value of $-\mathrm{Re}\,\lambda$ among the eigenvalues of K. The hypocoercive index is then

$$r = \frac{\lambda_{\text{hyp}}}{\lambda_F} = \lambda_{\text{hyp}}.$$

Each draw of (A, J) yields a single value of r and a value of $\kappa(A)$.

8.2 Numerical inequalities and typical behaviour

For each pair $(d, \kappa_{\max})$ the script generates a large number of independent samples, computes λ_{hyp}, $\kappa(A)$ and r, and stores them in an `npz` file. The main quantities of interest are:

- the fraction of samples violating the inequalities $r \geq 1$ or $r \leq \kappa(A)$;
- the empirical distribution of r for fixed $\kappa(A)$ and d;
- the dependence of the typical value of r on d and $\kappa_{\max}$.

Across ensembles with dimensions d between two and eight and condition numbers up to $\kappa_{\max} \approx 10^3$, large scans show no violations of the bounds

$$1 \leq r \leq \kappa(A).$$

This is consistent with the geometric picture of Section 4: the invariant $r_\star(A, J)$ on a slow block is a planewise average of eigenvalues of A, hence must lie between the smallest and largest eigenvalues.

Moreover, for moderate condition numbers the distribution of r is sharply peaked near 1. If $\kappa(A)$ is of order unity, most symplectic planes E_j intersect the coldest eigendirections of A in a fairly isotropic fashion, and the average $\bar{d}_j$ is close to d_1. Only when $\kappa(A)$ is large and the eigenvalues of A are very anisotropic do values of r substantially greater than one appear, and even then most samples satisfy $r \ll \kappa(A)$.

From the UIH perspective this suggests that pure Fisher behaviour with $r \approx 1$ is structurally stable and generic for random metriplectic pairs, while large hypocoercive speedups require specific alignments between the Fisher eigenstructure and the symplectic planes of J. The explicit two field and GKLS examples constructed earlier are therefore not extreme outliers, but representative of a controlled way to realise intermediate values of $r_\star$ through anisotropic diffusion coefficients and simple reversible couplings.

9 Discussion and outlook

The main message of this paper is that once an information metric and an entropy clock are fixed, hypocoercive renormalisation acquires a simple geometric structure. The UIH framework singles out a universal form for generators, $K = G + J$, in which the symmetric part is a Fisher gradient flow and the skew part is a reversible circulation. On mean zero subspaces the Fisher gap λ_F defines a canonical timescale and the hypocoercive rate λ_{hyp} measures how much faster the full dynamics relaxes in the entropy geometry.

A single dimensionless number,

$$r = \frac{\lambda_{\text{hyp}}}{\lambda_F},$$

then compares the true irreversible rate to the bare diffusion scale. In simple hydrodynamic settings with two coupled fields this index is explicitly computable and independent of wavenumber, giving a first hint of universality. The Fisher symplectic analysis shows that in general slow blocks decompose into two dimensional planes on which the reversible sector acts as a rotation, and that the relevant invariant is the minimal average diffusion on such a plane. The bounds

$$1 \le r_\star(D, J_0) \le \kappa(D)$$

follow directly from this geometric picture.

Once this structure is in place, a natural class of renormalisation maps emerges. Fisher compatible coarse grainings act as partial isometries in the information metric, and entropy clock rescaling keeps the Fisher gap fixed at each step. On ideal slow blocks this reduces to an exact spectral projection that leaves the hypocoercive index unchanged. In translation invariant hydrodynamics it approximates a projection onto small wavenumbers, preserving the infrared invariant $r_\star(D, J_0)$. Coarse grainings that ignore the Fisher metric or locality, by contrast, do not respect this structure and generically destroy any simple notion of universality.

The nonlinear two field ring confirms that the index derived from the linearised generator remains dynamically meaningful beyond the linear regime: in the entropy clock, small perturbations around the homogeneous state decay at the predicted rate, with nonlinear advection modifying the shape of the transient but not the leading exponent. The explicit qubit GKLS model shows that the same universality class can be realised in a microscopic quantum system, while the qutrit multi charge example anchors the Fisher operator A in the standard BKM geometry of quantum information theory. The random ensemble scans demonstrate that the inequalities $1 \le r \le \kappa(A)$ are not curiosities of special models but generic features of metriplectic pairs, and that values $r \approx 1$ are typical when the Fisher metric is not highly anisotropic.

Placed alongside our previous work, these results suggest the following picture. The reversible part of UIH shows that a Fisher metric and a canonical Poisson structure single out Schrödinger dynamics as the unique reversible hydrodynamics. The entropy geometry and gravity work shows that the same metric data organise irreversible gradient flows and scalar Fisher gravity, with cost-entropy inequalities and curvature coercivity. The irreversible density paper identifies a common Fisher-Dirichlet operator underlying Markov, Fokker-Planck and GKLS density sectors and proves a finite dimensional hypocoercivity theorem for generators $K = G + J$.

The present paper adds renormalisation to this structure. It shows that when one coarse grains in a Fisher compatible way and measures time in the entropy clock, irreversible flows fall into simple universality classes labelled by Fisher symplectic invariants $r_\star$. One current blocks always flow to the pure Fisher class $r_\star = 1$, while multi current blocks flow to fixed points determined by the geometry of J in the Fisher metric. The simplest nontrivial class, with $r_\star = 3/2$, already appears in the two field ring and in a single qubit GKLS model.

There are several directions for future work.

First, on the classical side, it would be natural to extend the renormalisation analysis to more realistic hydrodynamic systems, such as coupled density and momentum fields in higher dimensions, and to explore whether Fisher compatible RG maps can be formulated for lattice discretisations of Navier-Stokes type equations. The role of non normal effects and the approach of $r(k)$ to $r_\star$ at small but finite wavenumber deserve a more systematic study.

Second, on the quantum side, one can consider spatially extended GKLS chains with local couplings and investigate whether Fisher compatible coarse grainings on the lattice of sites lead to RG flows of the kind described here, possibly in combination with Kähler-type RG flows on the state space as in the reversible UIH work. The interaction between block spin style coarse graining and information geometric flows could provide a bridge between UIH and more traditional real space RG methods in condensed matter.

Third, on the gravitational side, the scalar Fisher gravity sector already identified in the entropy geometry paper suggests that Fisher metrics on density fields can source effective gravitational potentials. It is natural to ask whether the hypocoercive indices $r_\star$ and entropy clocks introduced here have analogues in the dynamics of such Fisher halos, for example as renormalised relaxation rates or effective temperatures in coarse grained gravitational systems.

Finally, from a more abstract standpoint, the Fisher symplectic invariant $r_\star$ may have implications for the design and analysis of numerical schemes and model reductions in open quantum and stochastic systems. Knowing that a reduced model preserves the correct entropy clock and hypocoercive index could serve as a practical criterion for evaluating the quality of coarse grained descriptions.

The code archive in Appendix 10 provides a compact, reproducible set of numerical experiments that support the claims made here. Together with the other components of the UIH programme, it invites a view of irreversible dynamics in which information geometry, reversible circulation and renormalisation are facets of a single operator framework.

9.1 Finite dimensional entropic RG testbed

To complement the continuum constructions in this paper, Appendix 11 develops a finite dimensional Fisher manifold model of entropic RG and tests the UIH picture numerically. There we fix a Fisher operator A, construct random UIH generators $K = G + J$ with prescribed Fisher spectra, and implement the Fisher compatible RG map that projects onto slow Fisher eigenspaces. Three numerical scans show that non UIH perturbations contract in Frobenius norm under this RG, and that the hypocoercive index $r = \lambda_{\mathrm{hyp}}/\lambda_F$ behaves as an RG C function for the composite UV to IR flow, with r driven to the pure Fisher value $r = 1$ in the single current class. This finite dimensional testbed provides a simple, reproducible proxy for the blockwise RG flow of Markov, Fokker–Planck and GKLS generators studied in the main text, and supports the interpretation of UIH as an entropic RG attractor class.

10 Code archive

The numerical renormalisation experiments in the main text are supported by a small Python suite. This appendix summarises their roles and input-output structure.

Available via github: https://github.com/feuras/uih_grav/

and

Zenodo: https://zenodo.org/records/17701239 - DOI: 10.5281/zenodo.17701238

`01_rg_rho_u_ring_linear_modes.py` Linear two-field ring dispersion and hypocoercivity index. The script constructs the discrete Laplacian L and central derivative D on a periodic ring of N sites for a coupled density-velocity system (ρ, u) with diffusivities $D_\rho, D_u > 0$ and reversible coupling $c > 0$. In Fourier space each wavenumber k yields a 2×2 block $K_k = G_k + J_k$, from which the script computes the Fisher gap $\lambda_F(k)$ from the symmetric part $-G_k$, the hypocoercive rate $\lambda_{\text{hyp}}(k)$ from the spectrum of K_k, and the entropy-clock index $r(k) = \lambda_{\text{hyp}}(k)/\lambda_F(k)$. The output `01_rg_rho_u_ring_linear_modes_output.npz` contains the mode labels, $\lambda_F(k)$, $\lambda_{\text{hyp}}(k)$ and $r(k)$ together with the theoretical prediction $r_\star(k \to 0) = (D_\rho + D_u)/(2\min(D_\rho, D_u))$. Its role is to give a clean, exactly solvable benchmark of the two-field universality class and to calibrate the theoretical entropy-clock index against explicit spectra.

`02_rg_rho_u_ring_block_vs_random.py` Good versus bad coarse-graining on the linear ring. Starting from the same linear (ρ, u) generator on a ring of length N_0, this script compares a UIH-compatible block RG with a deliberately "bad" random decimation. At each RG step it halves the number of sites, constructs the coarse-grained generator $K' = G' + J'$ by either block-averaging (good RG) or random projection (bad RG), and rescales time so that the Fisher gap of the good scheme remains fixed. For both flows it tracks the Fisher gap λ_F, the hypocoercive rate λ_{hyp} and the index r. The output `02_rg_rho_u_ring_block_vs_random_output.npz` records the sequence of lattice sizes, gaps and indices. Numerically one finds that good RG preserves λ_F and the target index $r \approx 3/2$ across scales, whereas bad RG drives λ_{hyp} and r away from their microscopic values. This provides a minimal demonstration that UIH-style coarse-graining singles out a stable entropy-clock universality class.

`03_rg_rho_u_ring_nonlinear_decay.py` Nonlinear two-field ring and slow-mode entropy clock. This flagship script promotes the linear (ρ, u) ring to a weakly nonlinear PDE system

$$\partial_t \rho = D_\rho L\rho - cDu - \alpha D(\rho u), \qquad \partial_t u = D_u Lu - cD\rho - \beta D(u^2/2),$$

on a periodic lattice, with small nonlinear couplings α, β. It first builds the linear generator $K = G + J$, computes λ_F, λ_{hyp} and the theoretical index $r_\star = (D_\rho + D_u)/(2\min(D_\rho, D_u))$, and extracts the slow hypocoercive eigenmode v_{slow} of K. The initial condition is chosen as a tiny perturbation along v_{slow} around the homogeneous state $(\rho, u) = (\rho_0, 0)$. The script then integrates the full nonlinear dynamics with an explicit RK4 scheme up to time T_{final}, sampling a Fisher-type norm

$$\|x\|_F^2 = D_\rho\, \delta\rho^\top (-L)\, \delta\rho + D_u\, \delta u^\top (-L)\, \delta u,$$

with mean-subtracted fields $\delta\rho, \delta u$. A log-linear fit of $\log \|x(t)\|_F$ over the window $[0.2T_{\text{final}}, 0.8T_{\text{final}}]$ yields an empirical decay rate λ_{fit}. The output `03_rg_rho_u_ring_nonlinear_decay_output.npz` stores parameters, time series, λ_F, λ_{hyp}, the theoretical index $r_\star$ and λ_{fit}. For the default near-linear regime one finds $\lambda_{\text{fit}}/\lambda_{\text{hyp}} \approx 1$ at the 10^{-3} level, showing that the linear entropy-clock rate survives weak nonlinear advection when initialised in the slow hypocoercive mode.

`04_rg_qubit_gkls_hypocoercivity.py` Qubit GKLS hypocoercivity in the Bloch picture. This script builds an explicit single-qubit GKLS generator with

Hamiltonian and noise

$$H = \frac{\Omega}{2}\,\sigma_z, \qquad L_x = \sqrt{\gamma_x}\,\sigma_x, \quad L_y = \sqrt{\gamma_y}\,\sigma_y.$$

It constructs the Bloch generator M via $M_{ik} = \frac{1}{2}\mathrm{Tr}(\sigma_i\,\mathcal{L}(\sigma_k))$ for $i, k \in \{x, y, z\}$, extracts the 2×2 x-y block K_{xy}, and decomposes $K_{xy} = G_{xy} + J_{xy}$ into symmetric and skew parts. From G_{xy} it computes the Fisher gap λ_F as the smallest eigenvalue of $-G_{xy}$; from the spectrum of K_{xy} it extracts the slow eigenvalue λ_{slow} and defines $\lambda_{\mathrm{hyp}} = -\mathrm{Re}\,\lambda_{\mathrm{slow}}$. The script then identifies the diagonal entries of $-G_{xy}$ as effective diffusivities D_x, D_y, forms the theoretical index $r_\star = (D_x + D_y)/(2\min(D_x, D_y))$, and compares it to $r = \lambda_{\mathrm{hyp}}/\lambda_F$. Finally, it integrates the Bloch dynamics in the x-y plane from an initial condition aligned with the slow eigenvector, tracks $\|r_{xy}(t)\|$, and fits a decay rate λ_{fit}. The output `04_rg_qubit_gkls_hypocoercivity_output.npz` contains M, K_{xy}, G_{xy}, J_{xy}, the spectral data and the time series. For the default parameters one finds $r = r_\star = 3/2$ exactly and $\lambda_{\mathrm{fit}}/\lambda_{\mathrm{hyp}} \approx 0.999$, placing this microscopic open quantum system cleanly in the same universality class as the two-field ring.

`05_rg_random_GJ_hypocoercivity_scan.py` Random finite-dimensional $K = G+J$ universality scan. This script samples an ensemble of real $d\times d$ generators $K = G+J$ with: (i) a symmetric negative definite part G constructed by drawing a random orthogonal matrix Q and a diagonal spectrum $\{d_i\}$ for $-G = Q\,\mathrm{diag}(d_i)\,Q^\top$, with d_i log-uniform in $[1, \kappa_{\max}]$ and rescaled so $\min d_i = 1$; and (ii) a skew part J drawn as a Gaussian random matrix antisymmetrised and scaled by a factor J_{scale}. For each sample the script computes the Fisher gap λ_F and condition number $\kappa = \lambda_{\max}/\lambda_F$ of $-G$, the hypocoercive rate λ_{hyp} from the spectrum of K, and the index $r = \lambda_{\mathrm{hyp}}/\lambda_F$. The output `05_rg_random_GJ_hypocoercivity_scan_output.npz` stores λ_F, $\lambda_{\max}$, κ, λ_{hyp} and r across the ensemble, together with simple counts of samples violating the inequalities $r \geq 1$ or $r \leq \kappa$. Large runs (e.g. $d = 6$, 2.5×10^5 samples) show numerically that $1 \leq r \ll \kappa$ is a robust feature of generic metriplectic pairs, not an artefact of special low-dimensional models.

`06_rg_multicharge_gkls_bkm_metric.py` Multi-charge qutrit GKLS with explicit BKM metric. This script constructs a three-level system with commuting "charges" H, Q_1, Q_2 diagonal in the computational basis, and a generalised Gibbs state $\rho_\star \propto \exp(-\beta H - \mu_1 Q_1 - \mu_2 Q_2)$. It then builds a GKLS generator with Hamiltonian part $-i[H, \cdot]$ and jump operators $L_{i\leftarrow j} = \sqrt{\Gamma_{i\leftarrow j}}|i\rangle\langle j|$ for all ordered pairs $i \neq j$, with rates $\Gamma_{i\leftarrow j}$ chosen to satisfy detailed balance with respect to $\rho_\star$. The resulting Liouvillian L is assembled as a 9×9 superoperator, and the script verifies stationarity by computing $\|L(\rho_\star)\|_F$. It then implements the full Bogoliubov-Kubo-Mori inner product at $\rho_\star$,

$$g_{\mathrm{BKM}}(A, B) = \sum_{i,j} c_{\mathrm{BKM}}(p_i, p_j)\,\tilde{A}_{ij}\,\tilde{B}_{ji}, \qquad c_{\mathrm{BKM}}(x, y) = \frac{\log x - \log y}{x - y},$$

for centred Hermitian directions A, B, where p_i are the eigenvalues of $\rho_\star$ and $\tilde{A}$ is A in the eigenbasis of $\rho_\star$. The script computes the BKM Gram matrix on the centred charges (H_c, Q_{1c}, Q_{2c}), compares it to the classical covariance matrix on the diagonal distributions (the Fisher matrix in parameter space), and extends the Gram matrix to a small Hermitian basis including off-diagonal operators X_{01}, Y_{01}. The output `06_rg_multicharge_gkls_bkm_metric_output.npz` contains $\rho_\star$, the superoperator, and the BKM matrices. This provides a compact, fully explicit example of a multi-charge GKLS model with a genuine quantum BKM metric, making the link between the UIH G-sector and the standard Kubo-Mori geometry concrete.

07_uih_rg_rIndex_pilot_scan.py Pilot finite-dimensional UIH hypocoercive index scan. This script implements a first "toy" entropic RG on UIH generators $K = G + J$ to test the behaviour of the index $r = \lambda_{\rm hyp}/\lambda_F$ under Fisher-compatible coarse graining. A Fisher operator $A = -G$ is sampled in dimension d_0 by drawing a random orthogonal matrix Q and log-uniform eigenvalues $\{\lambda_i\} \in [1, \kappa_{\max}]$, rescaled so that $\min \lambda_i = 1$. A random antisymmetric J is drawn from a Gaussian matrix antisymmetrised and scaled by a factor `j_scale`, and the UIH generator is set to $K = G + J$. The script then defines an "entropic RG" step by diagonalising A, projecting onto the slowest eigenvectors via a matrix P, and forming the coarse-grained triple $A' = P^\top AP$, $G' = P^\top GP$, $J' = P^\top JP$. For each sampled K, it performs a short RG ladder (e.g. $12 \to 8 \to 4$), computes the Fisher gap λ_F, the hypocoercive rate $\lambda_{\rm hyp}$, and the index r at each scale, and prints summary statistics (min, max, mean) and monotonicity counts $r_{k+1} > r_k$, $r_{k+1} < r_k$ across the ensemble. This provides a first numerical confirmation that in a generic UIH ensemble r tends to decrease under Fisher-compatible coarse graining and is driven towards the pure Fisher value $r = 1$.

08_uih_rIndex_multiscale_Cfunction_scan.py.py Multiscale UIH RG C-function scan for the hypocoercive index. This script extends the pilot scan to a large-scale, multi-level entropic RG flow, treating $r = \lambda_{\rm hyp}/\lambda_F$ as a candidate RG C-function. As in the previous script, a random Fisher operator $A = -G$ with log-uniform spectrum in $[1, \kappa_{\max}]$ and a random antisymmetric J (scaled by `j_scale`) define a UIH generator $K = G + J$ in dimension d_0. The script then applies a Fisher-compatible RG ladder through a long dimension chain, for example

$$32 \to 28 \to 24 \to 22 \to 20 \to 18 \to 16 \to 14 \to 12 \to 10 \to 8 \to 6 \to 4 \to 2 \to 1,$$

at each step diagonalising A, projecting onto the slowest eigenmodes and propagating G and J accordingly. For each sample and each RG level it computes λ_F, $\lambda_{\rm hyp}$ and r, and aggregates over a large ensemble (e.g. 10^5 samples) using a multiprocessing pool (up to 22 workers). The output consists of printed statistics for r at each scale (min, max, mean), stepwise monotonicity counts $r_{k+1} \gtrless r_k$, UV–IR comparisons $r_{\rm IR} \gtrless r_0$, and a correlation coefficient $\mathrm{corr}(r_0, r_{\rm IR} - r_0)$. Large runs show that while r can fluctuate at individual steps, the full UV→IR flow strictly drives every sample to $r_{\rm IR} = 1$, with $\mathrm{corr}(r_0, r_{\rm IR} - r_0) \approx -1$, providing strong evidence that r behaves as a genuine RG C-function on the UIH manifold under entropic RG.

09_uih_tg_attractor_error_contraction_scan.py.py UIH-attractor error contraction scan under Fisher RG. This script tests the "UIH as RG attractor" picture by explicitly adding a non-UIH perturbation and tracking how it contracts under entropic RG. For each trial it samples a UIH pair (A, G, J) in dimension d_0 as before, constructs the base generator $K = G + J$, and then draws a full Gaussian matrix X which is rescaled to define an "error" E_0 with $\|E_0\|_F =$ `e_scale` $\|K\|_F$ (typically `e_scale` = 0.5). The perturbed generator is $L_0 = K + E_0$. The script then applies the same Fisher-compatible RG ladder to (A, G, J, E), projecting all four objects via $P^\top(\cdot)P$ at each step. It computes two diagnostics at every RG level: the absolute error contraction $\|E_k\|_F/\|E_0\|_F$, and the relative error $\|E_k\|_F/\|K_k\|_F$ compared to the UIH part. Over a large ensemble (e.g. 10^5 samples, 22 workers) it prints summary statistics (min, max, mean) for both quantities at each dimension and monotonicity counts for $\|E_k\|_F/\|E_0\|_F$ across RG steps. The results show strict stepwise contraction of the absolute error norm (no samples with $\|E_{k+1}\|_F > \|E_k\|_F$) and an overall decay of $\|E_k\|_F$ proportional to $\dim_k/\dim_0$, together with bounded relative error that typically peaks at intermediate scales and shrinks again in the deep IR. This provides direct numerical evidence that Fisher-compatible RG is a contraction onto the UIH manifold, making the "UIH as entropic RG attractor" statement precise in finite dimension.

11 Finite dimensional entropic RG and UIH attractors

This appendix records a finite dimensional model of the entropic renormalisation group (RG) used in the main text, together with three numerical scans that support the claim that Universal Information Hydrodynamics (UIH) is an RG attractor class. The aim is to make precise, in the simplest setting, the following two statements.

1. For a fixed Fisher metric and entropy, entropic RG contracts any thermodynamically sane generator onto the UIH manifold of generators of the form $K = G + J$, with G the Fisher gradient operator and J metric skew.
2. Within that manifold, the hypocoercive index $r = \lambda_{\text{hyp}}/\lambda_F$ behaves as an RG monotone for the composite UV to IR flow, and in the single current class flows to the pure Fisher value $r = 1$.

The model and the simulations described here are implemented in the code archive scripts `07_uih_rg_rIndex_pilot_scan.py`, `08_uih_rIndex_multiscale_Cfunction_scan.py.py` and `09_uih_tg_attractor_error_contractio`

11.1 Finite dimensional Fisher manifold and UIH generators

Fix a real vector space $V \cong \mathbb{R}^n$ equipped with a Fisher inner product

$$\langle x, y\rangle_M \;=\; x^\top M y, \qquad M \text{ positive definite.}$$

For any linear operator $L\colon V \to V$ define its metric adjoint

$$L^\sharp := M^{-1} L^\top M,$$

and its symmetric and antisymmetric parts

$$S_L := \tfrac{1}{2}\,(L + L^\sharp), \qquad A_L := \tfrac{1}{2}\,(L - L^\sharp).$$

In the main text the Fisher gradient sector is encoded by a symmetric generator G whose Dirichlet form reproduces the Fisher quadratic form for the entropy and free energy. In the finite dimensional model we write

$$A := -G,$$

and choose an M orthonormal basis that diagonalises A,

$$Av_i = \lambda_i v_i, \qquad 0 < \lambda_1 \le \cdots \le \lambda_n,$$

so that $M = I$ and $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ in this basis.

A finite dimensional UIH generator is then taken to be

$$K = G + J, \qquad G = -A = G^\top, \qquad J = -J^\top,$$

exactly as in the K paper, where G encodes the Fisher gradient flow and J encodes metric compatible circulation.

The Fisher gap is

$$\lambda_F(K) := \lambda_1,$$

the smallest eigenvalue of A on the mean zero subspace. The hypocoercive decay rate of K is defined as

$$\lambda_{\mathrm{hyp}}(K) := \min\{-\mathrm{Re}(\mu) : \mu \text{ eigenvalue of } K,\ \mathrm{Re}(\mu) < 0\},$$

and the hypocoercive index is

$$r(K) := \frac{\lambda_{\mathrm{hyp}}(K)}{\lambda_F(K)}.$$

In the random ensemble used in the scans, A is constructed as

$$A = Q\,\mathrm{diag}(\lambda_1, \dots, \lambda_n)\,Q^\top,$$

with Q Haar distributed and the λ_i drawn log uniformly in $[1, \kappa_{\max}]$ then rescaled so that $\lambda_1 = 1$. The antisymmetric part J is drawn as

$$X \sim \mathcal{N}(0,1)^{n\times n}, \qquad J = \alpha_J (X - X^\top),$$

with a fixed scale α_J = `j_scale`. This is the finite dimensional avatar of a generic metriplectic UIH generator with a given Fisher sector and a thermally sane skew part.

11.2 Entropic RG as Fisher compatible projection

The entropic RG map used in the numerical experiments is defined purely in terms of the Fisher operator A. For a given target slow dimension $k < n$ one first diagonalises A,

$$A = V\,\mathrm{diag}(\lambda_1, \dots, \lambda_n)\,V^\top, \qquad V \in \mathrm{O}(n),$$

then selects the eigenvectors corresponding to the k smallest eigenvalues. Writing P for the $n \times k$ matrix whose columns are those eigenvectors, the Fisher compatible RG step is

$$A' = P^\top A P, \qquad G' = P^\top G P, \qquad J' = P^\top J P, \qquad K' = G' + J'.$$

By construction A' is the Fisher operator on the coarse grained slow space and $G' = -A'$. The skew part J' remains antisymmetric with respect to the induced metric. Thus if K is UIH then K' is again UIH on the slow block.

A multi step RG ladder is defined by a decreasing sequence of dimensions

$$n_0 > n_1 > \cdots > n_L,$$

with n_0 the UV dimension and n_L the IR target. Starting from an initial UIH generator K_0 with Fisher operator A_0, one constructs K_1 on dimension n_1 by projecting with the slow eigenspace of A_0 as above, then repeats the process at each scale, diagonalising the current Fisher operator A_k and projecting to dimension n_{k+1}. The scans in this appendix use ladders of the form

$$12 \to 8 \to 4,$$

for the pilot test, and

$$32 \to 28 \to 24 \to 22 \to 20 \to 18 \to 16 \to 14 \to 12 \to 10 \to 8 \to 6 \to 4 \to 2 \to 1,$$

for the multiscale runs.

In the full continuous RG story one also rescales time at each step so that the Fisher gap is kept fixed along the flow. Since this multiplies λ_F and λ_{hyp} by the same factor, the index r is invariant under such rescalings and they are omitted in the finite dimensional scans.

11.3 Analytic baseline: contraction towards the UIH manifold

Although the numerical scans involve random ensembles, the basic contraction mechanism of entropic RG can be seen directly at the level of linear algebra.

Let $\mathcal{M}_{\text{UIH}}$ denote the set of UIH generators with a fixed Fisher operator A and metric skew condition. Any generator L acting on V can be decomposed as

$$L = K + E, \qquad K \in \mathcal{M}_{\text{UIH}},$$

where E measures the non UIH part of L. In practice one can take K to be the unique generator with symmetric part $S_K = -A$ and antisymmetric part $A_K = A_L$ equal to the metric skew part of L. The distance to the UIH manifold is then measured in Frobenius norm,

$$d(L, \mathcal{M}_{\text{UIH}}) := \inf_{K \in \mathcal{M}_{\text{UIH}}} \|L - K\|_F, \qquad \|X\|_F^2 = \text{Tr}(X^\top X).$$

Given an RG projector P built from the slow eigenvectors of A as above, the coarse grained generator is

$$L' = P^\top L P = P^\top K P + P^\top E P.$$

The first term is again UIH on the slow space, with Fisher operator $A' = P^\top A P$. The projected error is

$$E' = P^\top E P.$$

Frobenius norm is non increasing under such orthogonal compression. In particular, since P has orthonormal columns and $\|P\|_{\mathrm{op}} = 1$, one has

$$\|E'\|_F \;=\; \|P^\top E P\|_F \;\leq\; \|E\|_F,$$

with equality only if the range and co-range of E are contained in the slow subspace selected by P. For a random full matrix E that event has measure zero. Thus, for generic error,

$$\|E'\|_F < \|E\|_F, \qquad d(L', \mathcal{M}'_{\mathrm{UIH}}) \leq \|E'\|_F < \|E\|_F \approx d(L, \mathcal{M}_{\mathrm{UIH}}).$$

Iterating along a multi step RG ladder produces a strict contraction towards the UIH manifold in Frobenius metric.

The finite dimensional scans summarised below confirm this contraction numerically in a large ensemble and quantify it very precisely.

11.4 Analytic baseline: $r \geq 1$ for UIH generators

For the hypocoercive index the key analytic observation in finite dimension is that any UIH generator satisfies a lower bound $r \geq 1$.

Work in the basis in which A is diagonal,

$$A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \qquad 0 < \lambda_1 \leq \cdots \leq \lambda_n.$$

By an orthogonal change of basis one can bring J into block diagonal form as a direct sum of real symplectic 2×2 blocks and possibly one dimensional zeros. On a two dimensional plane spanned by eigenvectors with eigenvalues λ_i, λ_j, the restriction of $K = -A + J$ can be written as

$$K_{ij} \;=\; \begin{pmatrix} -\lambda_i & \omega \\ -\omega & -\lambda_j \end{pmatrix}$$

for some real ω. The eigenvalues of this block are

$$\mu_\pm = -\tfrac{1}{2}(\lambda_i + \lambda_j) \pm \tfrac{1}{2}\sqrt{(\lambda_i - \lambda_j)^2 - 4\omega^2}.$$

If the discriminant is negative these form a complex conjugate pair. If it is non negative they are real. In either case the real parts of both eigenvalues are

$$\mathrm{Re}(\mu_\pm) = -\tfrac{1}{2}(\lambda_i + \lambda_j).$$

On a one dimensional block with no skew part one simply has eigenvalue $-\lambda_i$.

The hypocoercive decay rate is the smallest positive decay rate that appears among all blocks,

$$\lambda_{\mathrm{hyp}}(K) = \min\{\lambda_i,\ (\lambda_i + \lambda_j)/2\}.$$

Since every $\lambda_i \geq \lambda_1$ and every arithmetic mean $(\lambda_i + \lambda_j)/2$ is also at least λ_1, it follows that

$$\lambda_{\mathrm{hyp}}(K) \geq \lambda_1 = \lambda_F(K), \qquad r(K) \geq 1.$$

Equality holds if and only if there is no plane or line in which the decay rate falls strictly below λ_1. This happens in the trivial case $J = 0$, and can also occur for certain degenerate choices of A and J where the skew part only mixes isotropic blocks with equal Fisher eigenvalues.

This lower bound on r is the finite dimensional analogue of the hypocoercivity results in the main text: the J sector can at best accelerate relaxation relative to pure Fisher, never slow it below the Fisher gap.

11.5 Pilot scan: r under a short RG ladder

The pilot script `07_uih_rg_rIndex_pilot_scan.py` implements a first test of the behaviour of r under Fisher compatible RG on a modest dimensional ensemble.

For each trial, the script:

1. Samples A and J as described above, with dimension $d_0 = 12$, log uniform spectrum for A in $[1, \kappa_{\max}]$ and random antisymmetric J scaled by `j_scale`.
2. Forms $G = -A$ and $K = G + J$, then computes λ_F, λ_{hyp} and $r = \lambda_{\text{hyp}}/\lambda_F$.
3. Applies an RG step to dimension $d_1 = 8$ by diagonalising A and projecting onto the eight slowest eigenvectors, propagating G and J accordingly. The new generator K_1 and its index r_1 are computed.
4. Applies a second RG step to dimension $d_2 = 4$ and computes r_2.

Over a sizeable ensemble of random UIH generators the script collects basic statistics for (r_0, r_1, r_2), including minima, maxima, means and counts of samples with $r_{k+1} > r_k$ or $r_{k+1} < r_k$.

The outcome is that while a non zero fraction of individual ladders exhibit small upward fluctuations in r at a single step, the typical behaviour is a reduction of r as one projects to slower Fisher blocks, and the range of possible values shrinks. This suggested that the hypocoercive index behaves like an RG monotone in expectation, and motivated the larger multiscale scan.

11.6 Multiscale scan: r as an RG C function

The script `08_uih_rIndex_multiscale_Cfunction_scan.py.py` extends the pilot experiment to a long RG ladder and a much larger ensemble, in order to test whether r behaves as an RG C function in the sense of the full UV to IR flow.

The set up is as follows.

- Dimension ladder

$$n_0 = 32 \to 28 \to 24 \to 22 \to 20 \to 18 \to 16 \to 14 \to 12 \to 10 \to etc.$$

- Random UIH ensemble: for each sample, draw A and J in dimension 32 with log uniform eigenvalues for A in $[1, \kappa_{\max}]$, rescaled so $\lambda_1 = 1$, and random antisymmetric J scaled by a fixed `j_scale`. Construct $G = -A$ and $K_0 = G + J$.
- Entropic RG: apply Fisher compatible RG as described above to flow from dimension 32 down to dimension 1, tracking the Fisher operator and the UIH generator at each step.
- Hypocoercive index: at each RG level k compute $\lambda_F(K_k)$, $\lambda_{\text{hyp}}(K_k)$ and $r_k = \lambda_{\text{hyp}}(K_k)/\lambda_F(K_k)$.

The script uses a multiprocessing pool (up to 22 workers) to perform N independent RG trajectories, typically $N = 10^5$, and records:

- For each level k, the minimum, maximum and mean of r_k across the ensemble.
- For each step $k \to k+1$, counts of samples with $r_{k+1} > r_k$, $r_{k+1} < r_k$ and $r_{k+1} = r_k$.
- For the full UV to IR flow $0 \to L$, counts of samples with $r_L > r_0$ versus $r_L < r_0$.
- The empirical correlation $\operatorname{corr}(r_0, r_L - r_0)$.

In a representative run with 10^5 samples the following qualitative behaviour is observed.

1. The mean of r_k decreases smoothly along the ladder, from values of order two at dimension thirty two down to values very close to one at dimension two and exactly one at dimension one. The distribution narrows as one descends, with the minimum approaching one from above and the maximum shrinking.
2. At each individual step a minority of trajectories exhibit a small increase in r between k and $k+1$, with the fraction of such events typically between ten and twenty per cent and decreasing towards the IR. The majority exhibit a decrease.
3. For the full UV to IR flow, every single sample satisfies $r_L < r_0$ when the IR dimension is one. The empirical correlation $\operatorname{corr}(r_0, r_L - r_0)$ is numerically equal to -1 within floating point precision.

The apparent strict monotonicity of r along the full ladder and the perfect linear anticorrelation between r_0 and $r_L - r_0$ have a simple analytic explanation. For the particular ladder used here, the final space has dimension one. The Fisher operator at the IR point is λ_1, the smallest eigenvalue of the UV A, and antisymmetry forces $J_L = 0$. The IR generator is therefore the pure Fisher generator

$$K_L = -\lambda_1, \qquad \lambda_F(K_L) = \lambda_{\text{hyp}}(K_L) = \lambda_1, \qquad r_L = 1$$

for every sample. Combining this with the analytic bound $r_0 \geq 1$ for UIH generators yields

$$r_L = 1 \leq r_0$$

with equality only if the UV generator was already on the pure Fisher face in an appropriate sense. The empirical result that $r_L < r_0$ for all samples in the random ensemble reflects the fact that the initial draw of J almost never lands on the degenerate pure Fisher manifold.

Thus, for this finite dimensional model and this RG ladder, the hypocoercive index r is a genuine C function for the composite UV to IR flow: it never increases and generically decreases, with a unique IR fixed value $r = 1$. The fact that r can fluctuate at intermediate steps but is strictly monotone for the full projection is quite typical of approximate RG monotones in statistical mechanics.

11.7 Non UIH perturbations and contraction onto the UIH manifold

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The script `09_uih_tg_attractor_error_contraction_scan.py.py` tests the contraction of non UIH perturbations under entropic RG, making the statement "UIH is an RG attractor" more quantitative.

For each trial, the script performs the following steps.

1. Sample a finite dimensional UIH generator $K_0 = G_0 + J_0$ as before in dimension $n_0 = 32$, with Fisher operator A_0, Fisher gap $\lambda_F(K_0)$ and index r_0.
2. Draw a full Gaussian matrix X and rescale it to define an "error" matrix E_0 with prescribed size relative to K_0, for example

$$\|E_0\|_F = e_{\text{scale}} \|K_0\|_F, \qquad e_{\text{scale}} = 0.5.$$

3. Define the perturbed generator $L_0 = K_0 + E_0$. Decompose L_0 as "UIH part plus error" via this construction.
4. Apply the same entropic RG ladder to the quadruple (A, G, J, E), projecting

$$A_{k+1} = P_k^\top A_k P_k, \quad G_{k+1} = P_k^\top G_k P_k, \quad J_{k+1} = P_k^\top J_k P_k, \quad E_{k+1} = P_k^\top E_k P_k$$

   at each step, where P_k projects onto the slow Fisher eigenspace of A_k, and $K_k = G_k + J_k$.
5. At each RG level k compute two diagnostics:

$$e_k^{\text{abs}} := \frac{\|E_k\|_F}{\|E_0\|_F}, \qquad e_k^{\text{rel}} := \frac{\|E_k\|_F}{\|K_k\|_F}.$$

Over an ensemble of N samples (again typically 10^5, parallelised over 22 workers) the script records for each level the minimum, maximum and mean of both e_k^{abs} and e_k^{rel}, as well as monotonicity counts for e_k^{abs} between successive RG steps.

The results are very simple.

Absolute contraction. For the absolute error norm $\|E_k\|_F/\|E_0\|_F$ one finds that:

- At the UV level $k = 0$, $\|E_0\|_F/\|E_0\|_F = 1$ by construction.
- At each subsequent RG step, every single sample satisfies

$$\|E_{k+1}\|_F < \|E_k\|_F,$$

  so that the monotonicity counts show zero "up" events and one hundred per cent "down" events across the ensemble.
- The mean value of $\|E_k\|_F/\|E_0\|_F$ follows the very simple law

$$\mathbb{E}\big[\|E_k\|_F/\|E_0\|_F\big] \approx \frac{n_k}{n_0},$$

  where n_k is the dimension at level k and $n_0 = 32$ is the UV dimension. For example the mean error norm is close to 0.5 at dimension sixteen and close to 0.125 at dimension four.

This behaviour is precisely what one would expect for a random full matrix under orthogonal projection. If E has i.i.d. entries of variance σ^2, then $E' = P^\top E P$ has expected Frobenius norm squared

$$\mathbb{E}\|E'\|_F^2 = \frac{n_k^2}{n_0^2}\,\mathbb{E}\|E\|_F^2,$$

so that the expected norm scales by a factor n_k/n_0. The numerical scan confirms that this basic linear algebra mechanism is exactly what the entropic RG implements on the non UIH error.

Relative error. For the relative size $\|E_k\|_F/\|K_k\|_F$ the behaviour is more structured. The mean starts at $\|E_0\|_F/\|K_0\|_F = 0.5$ by construction, then rises to values of order one at intermediate scales and eventually falls again in the deep IR. In a typical run the mean relative error grows to around unity at dimensions of order ten, peaks modestly above one at intermediate scales, and then drops back towards order one as the dimension approaches one.

This is consistent with the picture that the UIH part K_k is better aligned with the slow Fisher eigenspaces than a random error, so the RG projection preserves more of K_k than of E_k. At the same time the UIH part itself becomes effectively lower dimensional as one approaches the slowest Fisher modes, so a residual error of fixed absolute size can temporarily be comparable to or larger than $\|K_k\|_F$ at intermediate scales before both decay absolutely in the deep IR.

It is important to note that in this script the UIH decomposition is not re fitted at each RG step. One propagates K_k and E_k separately by projection, rather than recomputing the best UIH approximation to $L_k = K_k + E_k$ at each scale. In other words, $\|E_k\|_F$ is an upper bound on the true distance from L_k to the UIH manifold at that scale. If one re fitted the symmetric part to the Fisher operator and absorbed skew error into the antisymmetric sector at each step, the actual distance to $\mathcal{M}_{\mathrm{UIH}}$ would decrease even faster.

11.8 Summary and relation to the main RG construction

The finite dimensional model in this appendix provides a clean local picture of the entropic RG and UIH attractor story developed in the main text.

- On a Fisher manifold with fixed Fisher operator A, the space of thermodynamically sane generators decomposes as an invariant UIH manifold $\mathcal{M}_{\mathrm{UIH}}$ plus error directions that violate Fisher compatibility or metric skew structure. The entropic RG map that projects onto slow Fisher modes acts as a strict contraction on those error directions and therefore attracts generic generators into $\mathcal{M}_{\mathrm{UIH}}$.
- Within $\mathcal{M}_{\mathrm{UIH}}$, the hypocoercive index $r = \lambda_{\mathrm{hyp}}/\lambda_F$ satisfies $r \geq 1$ and, for the composite UV to IR flow considered here, is strictly decreasing along the RG trajectory, with a unique IR fixed point $r = 1$ in the single current class. The numerical scans show that in a large random ensemble every sample flows to this pure Fisher value under Fisher compatible RG, and that the net change $r_{\mathrm{IR}} - r_0$ is linearly anticorrelated with the UV value r_0.

Together, these results support the interpretation of UIH as an entropic RG attractor class. In the continuum setting of Markov chains, Fokker–Planck operators and GKLS dynamics treated in the main body of the RG paper, the same algebraic structure appears blockwise on each slow sector. The finite dimensional analysis here can be read as the model computation on each such block: non UIH corrections are RG irrelevant in the Fisher compatible sense, and the remaining degrees of freedom are exhausted by a small number of UIH invariants. In particular, in the single current case the only dynamical scalar that survives in the IR is the scale free index r, which flows to one, while the Fisher gap λ_F sets the timescale and the scalar Fisher curvature sector $\mathcal{K}_F$ encodes the geometry.

The three scripts:

```
07_uih_rg_rIndex_pilot_scan.py
08_uih_rIndex_multiscale_Cfunction_scan.py.py
09_uih_tg_attractor_error_contraction_scan.py.py
```

provide compact, reproducible demonstrations of these claims in the simplest possible finite dimensional setting.

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