The Converse Madelung Answer
Quantum Hydrodynamics and Fisher Information Geometry

J. R. Dunkley

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Abstract

We study irreversible response for coarse grained densities in Fisher-regularised quantum hydrodynamics, working within a local metriplectic framework. The state space, boundary class and a uniformly elliptic symmetric mobility G are fixed once and for all, and all constructions take place in the weighted $H_\rho^{-1}(G)$ geometry. Three instantaneous objects are singled out: the realised irreversible drift generated by G, a cost-entropy inequality that links control cost to entropy production, and a curvature coercivity bound on the Fisher functional. All three are invariant under the addition of any reversible drift generated by an antisymmetric operator J satisfying a weighted Liouville constraint. Equality in the cost-entropy bound picks out a one dimensional irreversible ray, and a simple "equality dial" quantifies the reversible content of a given evolution. Assumptions are minimal, convex free energy, strictly positive densities, symmetric uniformly elliptic G, and the H_ρ^{-1} tangent model. All identities are supported by operational diagnostics and reproducible code. Read together with the companion paper, the present results identify the dissipative metriplectic channel compatible with the same Fisher geometry and Wasserstein-Otto tangent. The combined picture gives a minimal reversible-irreversible split for Fisher-regularised quantum hydrodynamics: the reversible current is the Fisher selected Schrödinger flow, while the present work fixes the local irreversible geometry and its equality and curvature certificates. A final scalar Fisher sector shows that the same weighted operator $L_{\rho,G}$ and Fisher quadratic form support a log density potential with a controlled Newtonian limit and a simple coupling to the Madelung Hamilton-Jacobi equation, kept deliberately within a scalar, weak field regime. All claims are necessity statements inside the stated axioms; no uniqueness of G or of the scalar dynamics is asserted beyond this local setting. The intended scope is quantum hydrodynamics and quantum information geometry for Fisher regularised Schrödinger dynamics with metriplectic dissipation.

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1 Introduction

Previously, our work on *The Converse Madelung Question* [1] treated the reversible side of hydrodynamic evolution by classifying the canonical bracket on (ρ, S), identifying the Fisher curvature that selects a linear unitary completion, and supplying operational verification within a minimal axiomatic class.

Here we provide the dissipative counterpart; again, in a specific axiomatic setting. We remain at the density level and characterise the no-work reversible cone via the weighted Liouville form without asserting a full Jacobi structure on densities.

We start with axioms that fix the state space and boundary classes, impose a local quadratic irreversible power with symmetric positive G, and require a reversible class that performs no work on F [2–5]. Calculus on the space of densities is taken in the H_ρ^{-1} tangent of the Wasserstein-Otto geometry [6–9].

Within this scope we ask how three natural scalars relate at a fixed state: the entropy production $\dot{\sigma}$, the smallest displacement curvature $\kappa_{\min}$, and the minimal quadratic cost $\mathcal{C}_{\min}$ to impress a given tangent v.

We show a sharp cost-entropy inequality that becomes an equality exactly on the gradient flow ray selected by the axioms, a curvature coercivity estimate with constants governed only by ellipticity of G and the positivity margin of ρ, and invariance of these scalars under the addition of any reversible drift. These same objects later serve as instantaneous readers in the assembled reversible dissipative picture and in the Fisher scalar sector, where they control a weak-field Newtonian limit and a simple Madelung coupling.

These are necessity statements inside the axioms, not equivalences beyond them. We do not assert uniqueness of G outside local ellipticity or any global identification across models.

Numerically we keep hypotheses visible through reproducible scripts. Instantaneous statements concern scalars defined at a fixed state ρ; path-integrated checks (such as $\int \dot{\sigma}\, dt = \Delta F$) are sanity tests of the plumbing and are reported separately.

A code archive (Appendix F) certifies the equality case under refinement, checks conservative plumbing and no work, measures a coarse-graining commutator that follows an ℓ^2 law, and recovers the quadratic action of G from probe Gram matrices.

Further testing adds an evolution variational inequality (EVI) probe with saturation on the irreversible ray, a Liouville no-work sweep, an H_ρ^{-1} orthogonality readout, tomography of G using scalar maps with cross-state checks, and a single-axiom failure table where symmetry, locality, positivity, or the tangent model is broken on purpose. Alignment identities report $\mathcal{R} = \cos^2\theta$ and explain when near equalities are observed. Path-integrated entropy is tested against ΔF and is insensitive to reversible drift at fixed targets.

Connections to thermodynamic geometry and optimal transport provide context for cost-entropy relations beyond linear response [12], and discrete or quantum analogues illustrate how changing the tangent model alters curvature-entropy structure [10, 11]. Our contribution is to place the dissipative assertions as necessities inside a minimal metriplectic setting with explicit certificates and falsifiers, complementing the reversible analysis of the companion paper and aligning the irreversible geometry with the Fisher structure that underlies the Schrödinger sector.

Later sections assemble the symmetric and antisymmetric blocks in a local setting and record simple operational checks, including a linear response reader obeying Kramers-Kronig within scope, a holonomy probe that is coarse-graining invariant on the tested family, and a Fisher scalar sector with a controlled Newtonian limit and a simple coupling to the Madelung dynamics.

All statements are necessity results *within* the stated axioms, ellipticity of G, and the positivity margin of ρ. We do not assert uniqueness of G or J beyond this local metriplectic setting.

At a glance, the three statements above are witnessed by a sharp equality on the gradient-flow ray, a curvature floor on the H_ρ^{-1} unit sphere, and invariance under reversible J; see Propositions 2.2, 2.3 and Appendix G. For a quantum reader, the main role of the present paper is to fix the irreversible metric, geometry and diagnostics that are compatible with the Fisher-Schrödinger structure of the companion work, without changing the underlying information geometry.

Relation to the reversible study. This work complements the reversible analysis developed in *The Converse Madelung Question*, which treated the canonical bracket on (ρ, S) and the role of Fisher curvature in establishing linearity after complexification. Within the stated axioms that paper identifies the Fisher functional as the unique reversible regulariser that supports a linear Schrödinger completion; here we remain entirely within the dissipative channel, fixing the local metriplectic form of the irreversible power and isolating the equality and curvature relations that persist when the reversible content is stripped away. Read together, the two papers form a minimal reversible dissipative pair under the same information-geometric conventions, differing only in the sign structure of the generator. The Fisher scalar sector and its Madelung coupling in Sec. 7 are recorded as an example of an emergent scalar slice built from the same density, the same weighted operator, and the same Fisher quadratic forms. For clarity, the κ-fixing and factorised-data arguments belong entirely to the reversible companion study and are not revisited here.

1.1 Reader Roadmap

Road map

Sections 2 and 3 establish the axioms, the cost-entropy inequality, curvature coercivity, and the metriplectic split on ρ.

Sections 4 and 5 give a didactic torus example together with numerical diagnostics (equality dial, coarse-grain commutator, reversible sweep), including typical failure modes when axioms are broken.

Sections 6 and 7 assemble the irreversible sector with the reversible Fisher-Schrödinger structure developed in the companion paper. Section 7 presents optional scalar and analogues within the same information geometry.

1.2 Results at a glance

Within the stated axioms (local ellipticity of G, positivity margin of ρ, admissible boundaries), we record three instantaneous statements at a fixed state ρ:

1. *Cost-entropy inequality (equality on the gradient-flow ray).* There is a sharp instantaneous inequality $\dot{\sigma}^2 \leq 2\, P_{\text{irr}}\, I_F$ that becomes an equality precisely on the gradient-flow direction selected by the axioms (Sec. 3.1).
2. *Curvature coercivity on the H_ρ^{-1} unit sphere.* The Hessian is controlled below by a curvature constant whose size depends only on ellipticity of G and the positivity margin of ρ (Sec. 3.2).
3. *Reversible invariance.* All three scalars above are invariant under adding any reversible J with the Liouville property (Sec. 2.4).

Falsifiers in Sec. G show that these identities fail once the geometry is altered (for example changing G or the boundary class).

Object	**Role**	**Where**
Cost-entropy scalar	Sharp inequality; equality on gradient-flow ray	Sec. 3.1
Curvature constant	Coercive Hessian bound on H_ρ^{-1} unit sphere	Sec. 3.2
Reversible invariants	Unchanged under adding any Liouville J	Sec. 2.4

Operator summary

Weighted Poisson operators.

$$L_\rho\varphi := -\nabla\cdot(\rho\nabla\varphi), \qquad L_{\rho,G}\varphi := -\nabla\cdot(\rho G\nabla\varphi),$$

defining the weighted H_ρ^{-1} and $H_\rho^{-1}(G)$ geometries.
Mobilities. G is the symmetric, positive definite mobility of the irreversible sector; J is the antisymmetric mobility of the reversible sector, required to satisfy the weighted Liouville constraint $\nabla_i(\rho J^{ij}) = 0$.
Complex packaging. $K = G + iJ$ is a bookkeeping device for diagnostics involving orthogonal quadratures; it is not a new dynamical operator.
Skew diagnostic map. H is a fixed skew map used only in the complex-phase diagnostic M.
Chemical potential. The free energy F induces a chemical potential $\mu = \delta F/\delta\rho$. All irreversible dynamics are generated by $L_{\rho,G}$ acting on μ.

2 Axioms, setting, and necessity results

Notation and orientation

ρ	density with $\rho > 0$ on the domain, normalised to mass 1
$F[\rho]$	free energy; F_∞ its minimiser value under the stated boundary class
$G(\rho, x)$	symmetric positive definite mobility (local, elliptic with fixed bounds)
$J(\rho, x)$	reversible operator with Liouville property (mass preserving)
$L_\rho\phi$	$-\nabla\cdot(\rho\nabla\phi)$
$L_{\rho,G}\phi$	$-\nabla\cdot(\rho G\nabla\phi)$
$\langle u, v\rangle_{H_\rho^{-1}}$	inner product induced by L_ρ^{-1} on zero-mean fields
$\dot\sigma$	instantaneous entropy production

We work on the periodic box $\mathbb{T}_L$ (or the stated admissible boundary class), with ρ bounded away from zero and G uniformly elliptic. All operators and norms are taken with respect to this setting.

Weighted H_ρ^{-1} pairing. Given a conservative field v with zero mean, let ϕ solve the Poisson problem $L_\rho\phi = -v$ with mean-zero gauge. For w with potential ψ defined likewise, set

$$\langle v, w\rangle_{H_\rho^{-1}} := \int_\Omega \rho\,\nabla\phi\cdot\nabla\psi\,dx, \qquad \|v\|^2_{H_\rho^{-1}} = \langle v, v\rangle_{H_\rho^{-1}}.$$

At a fixed state, define $\dot\sigma(\rho) \equiv \int_\Omega \rho\,(\nabla\mu)^\top G(\nabla\mu)\,dx$. For the realised irreversible direction $v_{\rm irr} = -L_{\rho,G}\mu$, one then has $\dot\sigma(\rho) = -\langle v_{\rm irr}, \mu\rangle$.

We work on a domain and regularity class that preserves conservative form and makes all variational statements precise. The irreversible closure and the reversible class are then derived as necessities from a short axiom list, with scalar certificates that are reproduced by the code archive in Appendix F.

Boundary, coarse-graining, and no-work details are documented in Appendix A.

Regularity. The core metriplectic results of Sections 2-3 require only $\rho \in H^1$ with a positive margin and a bounded, uniformly elliptic G. Some asymptotic and commutator expansions in Appendix C assume additional smoothness (e.g. $\rho \in H^3$ and $G \in C^1$) and should be viewed as diagnostics within that smoother subclass.

Positivity. Strict positivity of ρ is treated as a standing hypothesis on the solution class. We do not attempt to prove positivity preservation for arbitrary free energies F and mobilities G; when the margin $\rho_{\min}$ collapses the Fisher geometry and the associated diagnostics are explicitly out of scope.

2.1 State space, free energy, and boundary classes

Let $\Omega \subset \mathbb{R}^d$ be either a periodic box or a bounded Lipschitz domain with outward unit normal n. We consider strictly positive densities

$$\rho \in H^1(\Omega), \qquad \rho_{\min} \equiv \operatorname*{ess\,inf}_{\Omega} \rho \geq \varepsilon > 0, \qquad \int_{\Omega} \rho \, dx = M > 0,$$

with boundary classes:

- periodic, or
- no-flux $j \cdot n = 0$ for the physical flux j,

as detailed in Appendix A. The free energy $F[\rho]$ is Fréchet differentiable on the positive cone and defines a chemical potential

$$\mu(\rho) \equiv \frac{\delta F}{\delta \rho} \quad \text{up to an additive constant.}$$

Only $\nabla\mu$ enters the dynamics and power balances. The weighted Poisson operator $\mathcal{L}_\rho \phi = -\nabla\cdot(\rho\nabla\phi)$ is symmetric, coercive on mean-zero H^1 functions, and induces the weighted H_ρ^{-1} pairing; see Appendix A.

All Karush-Kuhn-Tucker (KKT) relations are written with negative sign in front of operator and are solved on the mean-zero subspace.

We denote the ellipticity window of G by $0 < \gamma_{\min} \leq \xi^\top G(\rho, x)\xi \leq \gamma_{\max}$ and write $\kappa_{\min}(\rho)$ for the smallest Wasserstein displacement curvature at ρ; these constants appear in all bounds below.

We fix the weighted operator by $L_{\rho,G}\phi \equiv -\nabla\cdot(\rho\, G\nabla\phi)$, so the gradient-flow ray is $v_0 = -L_{\rho,G}\mu = \nabla\cdot(\rho\, G\nabla\mu)$. All KKT solves are written with a leading minus and carried out on the mean-zero subspace.

We take ρ to be a positive density on Ω with total mass $\int_\Omega \rho\, dx = M$. Depending on context ρ may be interpreted either as a probability density ($M = 1$) or as a coarse-grained mass density; all constructions depend only on ρ itself. Any dimensional conversion to a physical mass density ρ_{m} is absorbed into scalar constants such as κ in later sections.

2.2 Minimal axioms

We adopt the following minimal hypotheses.

A1 State and mass. The state is $\rho \in H^1(\Omega)$ with $\rho_{\min} \geq \varepsilon > 0$ and the evolution is conservative,

$$\partial_t \rho \;=\; -\nabla \cdot j, \qquad \int_\Omega \partial_t \rho \, dx = 0,$$

within the boundary classes of Section 2.1.

A2 Free energy and Lyapunov sign. There is a free energy $F[\rho]$ with $\mu = \delta F / \delta \rho$ such that along the irreversible channel $\dot{F} \leq 0$.

A3 Local quadratic dissipation. At fixed state ρ, the instantaneous irreversible power is a local quadratic form in the driving gradient,

$$P_{\mathrm{irr}}(\rho; \mu) \;=\; \frac{1}{2} \int_\Omega \rho \, (\nabla \mu) \cdot G(\rho, x) \, (\nabla \mu) \, dx,$$

with $G(\rho, x)$ bounded, symmetric, and strictly positive definite pointwise. No nonlocal kernels appear in P_{irr}.

A4 Probe locality and relabelling invariance. Small probe variations of μ are local and insensitive to smooth relabellings of coordinates within a homogeneous medium. In particular, the quadratic response that defines G is invariant under rigid translations and rotations on the periodic box.

A5 Steepest descent. Among all conservative directions $v = -\nabla \cdot j$ that achieve the same P_{irr} at fixed ρ, the realised irreversible direction maximises the instantaneous entropy production $\dot{\sigma} = -\langle v, \mu \rangle$. Equivalently, the realised flux is the KKT minimiser of power subject to the continuity constraint.

A6 Reversible no-work. The reversible channel performs no-work on F for any smooth μ, that is $P_{\mathrm{rev}}(\rho; \mu) = \int_\Omega \mu \, \partial_t \rho \big|_{\mathrm{rev}} \, dx = 0$.

> *Remark* (Scope test for A6). A nonzero reversible power $P_{\mathrm{rev}} \neq 0$ places a run outside the no-work cone at the current ρ. Within our scope, $P_{\mathrm{rev}} = 0$ holds if and only if the reversible class admits $J^\top = -J$ and the weighted

Liouville identity $\nabla\cdot(\rho J) = 0$. Instantaneous violations of the dial reflect a break of antisymmetry or of the weighted Liouville constraint, not a contradiction with the canonical bracket used in the companion reversible paper.

A7 Symmetries and boundary class. All statements are scoped to the boundary classes in Section 2.1. Any symmetry is within those classes only.

2.3 Necessity of the irreversible generator

The axioms above force the weighted H^{-1} geometry and the Onsager direction.

Proposition 2.1 (*Weighted H^{-1} tangent and identifiability*). *Under A1-A4, any conservative direction v at fixed ρ can be uniquely represented as $v = \nabla\cdot(\rho\nabla\phi)$. The power P_{irr} induces the norm*

$$\|v\|_{G,\rho}^2 \;=\; \int_\Omega \rho\,(\nabla\phi)\cdot G(\rho,x)\,(\nabla\phi)\,dx,$$

which is equivalent to the H_ρ^{-1} norm. Moreover, G is identifiable from small probe responses by the Gram matrix $B_{ij} = \int_\Omega \rho\,(\nabla\phi_i)\cdot G\,(\nabla\phi_j)\,dx$ on any separating set of probe potentials $\{\phi_i\}$.

Proposition 2.2 (*Onsager steepest descent and equality case*). *Under A1-A5, the unique irreversible direction at ρ is*

$$v_{\text{irr}} \;=\; \nabla\cdot\big(\rho\,G(\rho,x)\,\nabla\mu(\rho)\big).$$

It realises the sharp Cauchy-Schwarz equality

$$\langle v_{\text{irr}},\mu\rangle^2 \;=\; 2\,P_{\text{irr}}(\rho;\mu)\,\dot\sigma(\rho),$$

with $\dot\sigma(\rho) = \int_\Omega \rho\,(\nabla\mu)^\top G(\nabla\mu)\,dx \geq 0$. Any other conservative direction with the same power yields a strict inequality.

Comment. Equation (2.2) provides a scalar certificate that is reproduced in the code archive by a mesh refinement study with dealiased products and subspace-consistent pairings.

Alternatives that violate A3 or A5 break (2.2). See Section G and Appendix A. In the (ρ, G) metric this certificate is the identity $\mathcal{R}(\rho; v_{\text{irr}}) = \cos^2\theta_{\rho,G} = 1$ with $\mathcal{R}$ and $\theta_{\rho,G}$ defined in Lemma 3.4.

2.4 Reversible no-work and orthogonality

Reader's map for A6. (i) The no-work cone is characterised by $J^\top = -J$ and $\nabla\cdot(\rho J) = 0$ at the fixed ρ. (ii) Along reversible trajectories F is constant and the reversible class is H_ρ^{-1}-orthogonal to the irreversible cone. (iii) The instantaneous scalars $\dot\sigma(\rho)$, $\kappa_{\min}(\rho)$, and $\mathcal{C}_{\min}(\rho; v)$ are insensitive to J by definition at fixed ρ.

A6 fixes the structure of the reversible class and its orthogonality to the irreversible cone.

Proposition 2.3 (*Weighted Liouville form and no-work*). *Under A6, the reversible flux can be written as*

$$j_{\mathrm{rev}} \;=\; -\,\rho\, J(\rho,x)\,\nabla\mu,$$

with $J^\top = -J$ and the weighted Liouville identity $\nabla\cdot(\rho J) = 0$. Conversely, these two conditions imply $P_{\mathrm{rev}}(\rho;\mu) = 0$ for all smooth μ and any choice of constant gauge. See Appendix B.

Proposition 2.4 (*Metriplectic orthogonality*). *Let v_{irr} be as in Proposition 2.2 and $v_{\mathrm{rev}} = \nabla\cdot(\rho J\nabla\mu)$ satisfy Proposition 2.3. Then*

$$\langle v_{\mathrm{rev}}, v_{\mathrm{irr}}\rangle_{H_\rho^{-1}} \;=\; 0, \quad \textit{equivalently } \langle v_{\mathrm{rev}}, \phi\rangle = 0 \textit{ for any } \phi \textit{ with } -L_\rho\phi = v_{\mathrm{irr}}.$$

Thus the reversible class lies in the H_ρ^{-1} orthogonal complement of the irreversible cone. In particular, F is constant along reversible trajectories and strictly decreases along irreversible ones unless $\nabla\mu \equiv 0$.

2.5 Consequence

Theorem 2.5 (*Local metriplectic decomposition*). *In our axiomatic setting, the evolution of ρ admits a unique decomposition*

$$\partial_t\rho \;=\; \nabla\cdot(\rho\, G(\rho,x)\,\nabla\mu) \;+\; \nabla\cdot(\rho\, J(\rho,x)\,\nabla\mu),$$

where $G = G^\top \succ 0$ is local, $J^\top = -J$ satisfies $\nabla\cdot(\rho J) = 0$, the equality certificate (2.2) holds on the irreversible channel, and the reversible channel has $P_{\mathrm{rev}} = 0$ and is H_ρ^{-1}-orthogonal to the irreversible cone. The pair (G, J) is identifiable up to scalar invariants on the irreversible side and up to the Liouville gauge on the reversible side.

For clarity, the minimal implications of these axioms and short constructive proofs of necessity are summarised in Appendix D.

3 Main statements and proof routes

Proof sketches and the logical dependency chain from the axioms to the statements below are given in Appendix D.

We now record three statements. Each uses only the hypotheses in Section 2. Proof sketches are given immediately, with full details deferred to Appendix E. Alignment diagnostics and falsifiers appear later in Sections 6 and 7.

3.1 Cost-entropy inequality (equality on the gradient-flow ray)

Proposition 3.1 (***Cost-entropy inequality; equality on the gradient-flow ray***). *For any admissible u with $v = -\nabla\cdot(\rho u)$ one has*

$$\langle v,\mu\rangle^2 \le 2\,\mathcal{C}(u)\,\dot\sigma(\rho),$$

hence

$$\mathcal{C}_{\min}(\rho;v) \ge \frac{\langle v,\mu\rangle^2}{2\,\dot\sigma(\rho)}.$$

Equality holds if and only if u is collinear with $G\nabla\mu$, equivalently v is collinear with the gradient-flow direction $-L_{\rho,G}\mu$ where $L_{\rho,G}\phi \equiv -\nabla\cdot(\rho G\nabla\phi)$.

Idea of proof. Integration by parts gives $\langle v,\mu\rangle = -\int \rho u\cdot\nabla\mu\,dx = \langle u, G^{-1}G\nabla\mu\rangle_\rho$. Apply Cauchy-Schwarz in the G^{-1} metric to obtain (3.1). Minimise over u to obtain (3.1). Equality holds exactly when u is everywhere collinear with $G\nabla\mu$. Full details are standard and included in Appendix A. □

3.2 Curvature coercivity on the H_ρ^{-1} unit sphere

For all v in the Wasserstein tangent,

$$\langle \mathcal{H}_F(\rho)\,v,\,v\rangle \ge \kappa_{\min}(\rho)\,\|v\|^2_{H_\rho^{-1}}.$$

Corollary 3.2 . *By ellipticity, $2\,\mathcal{C}_{\min}(\rho;v) \in [\gamma_{\min},\gamma_{\max}]\,\|v\|^2_{H_\rho^{-1}}$, hence*

$$\langle \mathcal{H}_F(\rho)\,v,\,v\rangle \;\ge\; \frac{\kappa_{\min}(\rho)}{\gamma_{\max}}\cdot 2\,\mathcal{C}_{\min}(\rho;v).$$

Section G shows that near uniformity the measured relaxation rates satisfy $r_{\mathrm{fit}} \approx 2\,\kappa_{\min}$, consistent with the log-Sobolev and mode-wise curvature anchors used here.

The Rayleigh quotient definition of $\kappa_{\min}$ gives the theorem. The corollary follows from the norm equivalence between the (ρ,G) energy and the H_ρ^{-1}

norm with constants $\gamma_{\min}, \gamma_{\max}$. Details are given in Appendix.

Remark (Two load-bearing hypotheses). If G is not symmetric positive, the metric Cauchy-Schwarz used in (3.1) is invalid and $\mathcal{R}(\rho; v)$ can exceed one for some v at fixed ρ. If the tangent norm is not the weighted H_ρ^{-1} norm, the Rayleigh structure that yields (3.2) can fail. Both failures are demonstrated by the falsifier dials.

3.3 Invariance under reversible drift

Proposition 3.3 (*Instantaneous invariance under* J). *For fixed ρ, the scalars $\dot{\sigma}(\rho)$, $\kappa_{\min}(\rho)$ and $\mathcal{C}_{\min}(\rho; v)$ depend only on (ρ, G, F) and are unchanged by adding any antisymmetric J to the instantaneous splitting* (2.5).

Proof. Each scalar is defined at the fixed state using only the symmetric quadratic forms and the Wasserstein tangent. The reversible drift does not enter their definitions. □

3.4 Alignment and near-equalities

The inequality in Proposition 3.1 becomes an equality when v is exactly collinear with $-L_{\rho,G}\mu$. In practice near-equalities are observed when v has a small angle with this direction in the H_ρ^{-1} inner product, or when the soft curvature mode aligns with $-L_{\rho,G}\mu$. We quantify this below.

Lemma 3.4 (*Alignment identity in the* (ρ, G) *metric*). *Let ϕ solve $-L_{\rho,G}\phi = v$ with $L_{\rho,G}\phi \equiv -\nabla\cdot(\rho G \nabla\phi)$. Define the (ρ, G) inner product on vector fields by*

$$\langle a, b\rangle_{\rho,G} \equiv \int_\Omega \rho\, a^\top G\, b\, dx, \quad \|a\|_{\rho,G}^2 = \langle a, a\rangle_{\rho,G},$$

and the angle

$$\cos\theta_{\rho,G} \equiv \frac{\langle \nabla\phi, \nabla\mu\rangle_{\rho,G}}{\|\nabla\phi\|_{\rho,G}\, \|\nabla\mu\|_{\rho,G}}.$$

Then the diagnostic ratio satisfies the exact identity

$$\mathcal{R}(\rho; v) \equiv \frac{\langle v, \mu\rangle^2}{2\,\mathcal{C}_{\min}(\rho; v)\,\dot{\sigma}(\rho)} = \cos^2\theta_{\rho,G} \in [0, 1],$$

with $\mathcal{R} = 1$ iff $\nabla\phi$ is collinear with $\nabla\mu$ (equivalently $u^\star \parallel G\nabla\mu$).

Proof. For the minimiser $u^\star = G\nabla\phi$ (KKT) one has

$$\langle v, \mu\rangle = -\int_\Omega \rho\, u^\star \cdot \nabla\mu\, dx = -\langle \nabla\phi, \nabla\mu\rangle_{\rho,G}.$$

Moreover, $2\,\mathcal{C}_{\min}(\rho; v) = \|\nabla\phi\|^2_{\rho,G}$ and $\dot\sigma(\rho) = \|\nabla\mu\|^2_{\rho,G}$. Hence

$$\mathcal{R}(\rho; v) = \frac{\langle \nabla\phi, \nabla\mu\rangle^2_{\rho,G}}{\|\nabla\phi\|^2_{\rho,G}\,\|\nabla\mu\|^2_{\rho,G}} = \cos^2\theta_{\rho,G}.$$

□

> *Remark* (Intuition for Lemma 3.4). The KKT map $v \mapsto \phi$ solves $-L_{\rho,G}\phi = v$, so $u^\star = G\nabla\phi$ is the unique minimal control. In the (ρ, G) inner product one has $2\,\mathcal{C}_{\min}(\rho; v) = \|\nabla\phi\|^2_{\rho,G}$ and $\dot\sigma(\rho) = \|\nabla\mu\|^2_{\rho,G}$, while the power pairing is $\langle v, \mu\rangle = -\langle \nabla\phi, \nabla\mu\rangle_{\rho,G}$.
>
> The diagnostic ratio therefore becomes a squared cosine between the two vectors $\nabla\phi$ and $\nabla\mu$ in the same metric, with equality if and only if they are collinear. Near-equalities occur when the minimal control direction aligns with the thermodynamic force, equivalently when v is close to the gradient flow ray $-L_{\rho,G}\mu$.

Alignment diagnostic. Let $-L_{\rho,G}\phi = v$ on the mean-zero subspace. Define

$$\cos\theta_{\rho,G} := \frac{\langle \nabla\phi, \nabla\mu\rangle_{\rho,G}}{\|\nabla\phi\|_{\rho,G}\,\|\nabla\mu\|_{\rho,G}}, \qquad R := \cos^2\theta_{\rho,G} \in [0, 1].$$

Thus R is obtained from one Poisson solve and two inner products; near-equality events appear as $R \approx 1$.

3.5 Didactic worked example on a torus

We illustrate constants in a simple model that matches common numerical experiments.

> **Proposition 3.5 (*Smallest curvature for a uniform state*).** *Let $\Omega = \mathbb{T}^d$ with period 2π in each direction, $G = I$, and $F[\rho] = \int \rho \log \rho\, dx + \frac{\lambda}{2}\int |\nabla\rho|^2\, dx$. At a uniform state $\rho \equiv \rho_0 > 0$ the curvature spectrum by Fourier mode $k \in \mathbb{Z}^d \setminus \{0\}$ is*
>
> $$\kappa(k) = |k|^2 + \lambda\,\rho_0\,|k|^4,$$
>
> *hence $\kappa_{\min}(\rho_0) = 1 + \lambda\rho_0$ attained at the first nonzero shell $|k| = 1$.*

Idea of proof. On the $H^{-1}_{\rho_0}$ tangent, $\kappa(k)$ is the Rayleigh quotient $\kappa(k) =$

$\langle H_F(\rho_0)v_k, v_k\rangle/\|v_k\|^2_{H^{-1}_{\rho_0}}$ for $v_k = -\nabla\cdot(\rho_0\nabla\phi_k)$ with $\phi_k(x) = \cos(k\cdot x)$ or $\sin(k\cdot x)$. At $\rho = \rho_0$, a first variation gives $\delta\mu = \rho_1/\rho_0 - \lambda\,\Delta\rho_1$, yielding the stated eigenvalues.

The entropy part gives $|k|^2$ and the Fisher part produces $\lambda\rho_0|k|^4$. □

> *Remark.* Proposition 3.5 provides a clean anchor for curvature scales in numerical plots. On a side-length L torus, replace $|k|^2$ by $(2\pi/L)^2|k|^2$ and $|k|^4$ by $(2\pi/L)^4|k|^4$, so $\kappa_{\min}(\rho_0) = (2\pi/L)^2 + \lambda\,\rho_0\,(2\pi/L)^4$ on the first shell. The angle envelope diagnostic $R = \cos^2\theta$ is used below to compare measured directions with this spectrum. We draw the reference line $\kappa_{\min}(\rho_0) = 1+\lambda\rho_0$; measured points collapse to this line as $|\rho-\rho_0|/\rho_0 \to 0$.

4 Scope, guardrails, and failure modes

This section records short analytic sketches that explain where the statements hold and where they do not, together with explicit caveats.

4.1 Non-convex free energy

If F is not convex, the smallest Wasserstein Hessian eigenvalue can be negative and the coercivity bound (3.2) fails. This is a true limitation with our setting. In many models convexity or displacement convexity is available on relevant subsets [8, 9]. Our results do not extend beyond convex settings. For example, with $F[\rho] = \int(\rho\log\rho - a\,\rho^2)\,dx + \frac{\lambda}{2}\int|\nabla\rho|^2 dx$, the smallest Wasserstein Hessian eigenvalue becomes negative for sufficiently large a, so the Rayleigh coercivity fails.

4.2 Invariant under reversible drift

We include a coupled channel plumbing check with $J \neq 0$ in the evolution that generates the flow snapshots. The instantaneous scalars are always evaluated at a fixed state and depend only on (ρ, G, F), so they are unchanged by J as Proposition 3.3 states. The numerical runs confirm this invariance. This test is labelled as a sanity check of the pipeline rather than as a validation of the main inequalities. See the weighted Liouville identity below and Appendix G, "Path-entropy invariance under reversible drift", for the algebraic condition and a direct numerical check.

4.3 Strongly nonlocal functionals

If the second variation of F acts as a strongly nonlocal operator on the Wasserstein tangent, the Rayleigh quotient structure that defines $\kappa_{\min}$ can be altered. We do not treat such cases here. Alternative transport geometries are an active topic and include variants such as Hellinger Kantorovich; see for instance [8] for background pointers. Our falsifier B illustrates that even a simple change of tangent norm breaks the intended chain. Concretely, if the second variation is a pseudo-differential operator of negative order or is unbounded on the H_ρ^{-1} tangent, the minimiser of the Rayleigh quotient need not be representable as $v = -L_{\rho,G}\phi$, and the $\kappa_{\min}$ link breaks; see Appendix G, "Wrong tangent norm."

4.4 Degenerate metrics and loss of ellipticity

If G loses ellipticity, the constants in our inequalities blow up and the numerical operators lose conditioning. This is consistent with the role of symmetric positive Onsager operators in GENERIC and metriplectic evolutions [2–5]. Our estimates require uniform bounds $0 < \gamma_{\min} \le \xi^\top G \xi \le \gamma_{\max} < \infty$. In practice we report the associated rise in KKT iteration counts as $\gamma_{\max}/\gamma_{\min}$ grows, to calibrate conditioning.

4.5 Nodes and vanishing density

If $\rho_{\min} \to 0$ then the H_ρ^{-1} norm degenerates and integrations by parts need additional care. In the reversible setting this is discussed in the context of hydrodynamic variables and Fisher curvature in [1]. Here we keep a fixed positivity margin and report the degradation of constants as a function of $\rho_{\min}$. Uniqueness of the mean-zero KKT potential also fails in this limit, so orthogonality claims are interpreted only on the positive cone.

Falsifiers are not confirmations of algebra; they show that the identities fail once the geometry is altered. This mirrors the falsifier philosophy used for reversible uniqueness and superposition in *The Converse Madelung Question* [1].

4.6 Metric symmetry break

We perturb the metric by a small antisymmetric component $G_\varepsilon = G + A_\varepsilon$ with $A_\varepsilon^\top = -A_\varepsilon$, while keeping all other steps unchanged. Since G_ε is no longer symmetric positive, the metric Cauchy Schwarz that underpins (3.1) is invalid. Numerically we observe that for fixed states and random admissible v, the ratio $\mathcal{R}(\rho; v)$ exceeds one for some samples once ε passes a small threshold.

This falsifies the symmetric positive hypothesis and demonstrates that metric symmetry is load-bearing rather than decorative. Theoretical background for metric positivity in gradient flows and GENERIC is classical [2–5].

Weighted Liouville identity. If $J^\top = -J$ and $\nabla\cdot(\rho J) = 0$, then for any smooth state

$$\int_\Omega \mu\, \nabla\cdot(\rho J \nabla\mu)\, dx = -\int_\Omega \rho\, (\nabla\mu)^\top J\, \nabla\mu\, dx = 0.$$

We record this as a convenient sufficient condition ensuring the reversible flux performs no-work on F.

4.7 Wrong tangent norm

We replace the Wasserstein tangent norm H_ρ^{-1} by a massless H^{-1} norm, that is we solve $-\Delta\phi = v$ without the ρ weight and evaluate quadratic forms accordingly. The Rayleigh structure that yields (E) is then lost, and we observe consistent violations of the curvature coercivity bound (3.2) on the same states. This aligns with the role of the Otto metric in displacement convexity and curvature lower bounds [6–9]. Empirically we observe violations on a non-zero fraction of random admissible states; a representative counterexample and script are listed in Appendix G. Discrete and quantum analogues underscore that the correct tangent model is essential for entropy curvature relations [10, 11].

4.8 Positivity margin degradation

We lower the positivity margin by shrinking $\rho_{\min}$ while keeping the same discrete operators. The constants in our estimates depend on $\rho_{\min}$ through coercivity. Numerically the fitted bounds degrade in line with the predicted dependence and the solvers require more iterations to meet the same residual tolerance. This is expected and is reported explicitly so that readers can calibrate conditioning. We print iteration counts alongside bound fits so readers can see this dependence.

Spatially varying mobility $G(x)$

Setting. We repeated the commuting-triangle and κ-oracle tests with spatially varying mobility $G(x) = 1 + \alpha\cos(2\pi x/L)$ for $\alpha \in [0, 0.8]$. At fixed ρ and $\lambda = 0.10$, we measured $R = \langle v, \mu\rangle^2/(2C_{\min}\dot\sigma)$ and $\cos^2\theta$ for (i) the gradient-flow ray $v = -L_{\rho,G}\mu$ and (ii) a random admissible v.

At uniform $\rho_0 = 1/L$ we also checked the mode oracle $\kappa(k)$. *Observation.* For all α, the equality case $1 - R \simeq 2\times 10^{-4}$ on the ray and $|R - \cos^2\theta| \lesssim 10^{-12}$

for random v persist, with solver residuals $O(10^{-12})$. The κ-oracle remains exact to machine precision for $k = 1, 2, 3$, independent of α.

Interpretation. Within the stated tolerances, the commuting-triangle equality and the mode curvature $\kappa(k) = (2\pi/L)^2|k|^2 + \lambda\,\rho_0\,(2\pi/L)^4|k|^4$ remain unchanged under moderate inhomogeneity of the mobility; in particular, the first-shell value $\kappa_{\min} = (2\pi/L)^2 + \lambda\,\rho_0\,(2\pi/L)^4$ is unaffected by α.

This indicates that the numerical and analytical structure of the metric are stable for non-uniform media, without asserting further generality.

4.9 What is not claimed

We do not claim necessity or uniqueness of the metriplectic structure, nor do we derive G or the tangent norm from minimal axioms. We also do not assert universal proportionality between any pair of the three scalars. We prove a sharp identity on one ray, two global inequalities with controlled constants, and structural invariance under reversible drift, all within the stated hypotheses.

Brief consistency checks

The following three items report compact, theory-facing verifications that extend the identities used in the paper. They are framed at the level of definitions and measurable scalars. No new claims are made beyond those already proved. Each item states hypotheses, objects evaluated, the equality or invariance that is expected to hold, and the observed tolerance levels under a representative discretisation on the periodic one-dimensional torus $\mathbb{T}_L$.

A. Internal multiplet (two components). *Setting.* Let $\rho = (\rho_1, \rho_2)$ be two strictly positive components with $\int_{\mathbb{T}_L}(\rho_1 + \rho_2)\,dx = 1$. Take

$$F[\rho] \;=\; \sum_{i=1}^{2}\int_{\mathbb{T}_L} \rho_i \log \rho_i\,dx \;+\; \frac{\lambda}{2}\sum_{i=1}^{2}\int_{\mathbb{T}_L} |\partial_x \rho_i|^2\,dx, \qquad \mu_i \;=\; \frac{\delta F}{\delta \rho_i}\,.$$

Let G be the scalar mobility $G = \mathrm{Id}$ acting componentwise and define the symmetric operator $L_{\rho,G}$ by

$$L_{\rho,G}\phi \;=\; -\,\big(\partial_x(\rho_1\,\partial_x\phi_1),\, \partial_x(\rho_2\,\partial_x\phi_2)\big)$$

restricted to the mean-zero subspace.

For any admissible tangent $v = (v_1, v_2)$ with $\int v_i\,dx = 0$ define the minimal control cost via the KKT problem $-\,L_{\rho,G}\phi = v$,

$$2\,C_{\min}(\rho; v) \;=\; \sum_{i=1}^{2}\int_{\mathbb{T}_L} \rho_i\,|\partial_x\phi_i|^2\,dx \qquad \dot\sigma(\rho) \;=\; \sum_{i=1}^{2}\int_{\mathbb{T}_L} \rho_i\,|\partial_x\mu_i|^2\,dx$$

and the commuting angle by

$$\cos\theta \;=\; \frac{\sum_i \int \rho_i\, \partial_x\phi_i\, \partial_x\mu_i\, dx}{\left(\sum_i \int \rho_i\, |\partial_x\phi_i|^2\, dx\right)^{1/2} \left(\sum_i \int \rho_i\, |\partial_x\mu_i|^2\, dx\right)^{1/2}} \qquad R(\rho;v) \;=\; \frac{\langle v,\mu\rangle^2}{2\, C_{\min}(\rho;v)\, \dot\sigma(\rho)}\,.$$

Verification. On the gradient-flow ray $v = -\,L_{\rho,G}\mu$ one expects alignment and equality $R \simeq 1$. For arbitrary admissible v one expects the identity $R = \cos^2\theta$.

Numerically, on $\mathbb{T}_L$ with $L = 40$, $N \in \{256, 512\}$, rectangle rule for integrals, pseudospectral derivatives with 2/3 de-aliasing, mean zero projection, and a small SPD stabiliser in $L_{\rho,G}$.

We observe $1-R \lesssim 10^{-3}$ on the ray and $|R-\cos^2\theta| \lesssim 10^{-12}$ for random v, with linear solves reaching residuals $O(10^{-12})$. This supports that the cost-entropy equality and the angle identity extend to a minimal internal multiplet within the stated hypotheses.

B. Static gauge covariance at the level of instantaneous scalars. *Setting.* Let $A(x)$ be a fixed $U(1)$ potential and $S(x)$ a phase. Form the covariant momentum $p = \partial_x S - qA$ and the associated reversible tangent $v_J = \partial_x(\rho\, p)$ at a fixed positive scalar density ρ with the same F and G as above. The metriplectic scalars at the frozen state are

$$\dot\sigma(\rho) \;=\; \int_{\mathbb{T}_L} \rho\, |\partial_x\mu|^2\, dx \qquad \mathcal{C}_{\min}(\rho;v) \text{ from } L_{\rho,G}\phi = v \qquad R(\rho;v) \;=\; \frac{\langle v,\mu\rangle^2}{2\,\mathcal{C}_{\min}\dot\sigma}\,.$$

Verification. Since $\dot\sigma$ and $\mathcal{C}_{\min}$ depend on (ρ, G, F) and the chosen tangent v but not on S or A independently of v, one expects: (i) $\dot\sigma(\rho)$ is unchanged under replacements $(A,S) \mapsto (\tilde A, \tilde S)$ at fixed ρ, (ii) for $v = v_J$ built from different static (A,S), $C_{\min}(\rho;v)$ varies with v as it should, and the commuting identity $R = \cos^2\theta$ continues to hold.

On the same discretisation as above we observe no change in $\dot\sigma(\rho)$ to numerical floor across many draws of (A,S), and $|R-\cos^2\theta| \lesssim 10^{-12}$ for each v_J. As an external check, the heat oracle with initial mode $\rho(x,0) = \rho_0[1+\varepsilon\cos(kx)]$ satisfies $\Delta F = \int_0^T \dot\sigma\, dt$ within $O(10^{-3})$ at $N = 4095$ in the presence of a static $A(x)$, consistent with the continuum identity.

These observations indicate that the instantaneous metriplectic scalars used in the paper are consistent with static gauge structure at the level of the stated hypotheses.

C. Controlled reversible-dissipative crossover at fixed state. *Setting.* At a fixed strictly positive ρ, define the pure gradient direction $v_G = -\,L_{\rho,G}\mu$ and a reversible proxy $v_J = \partial_x(\rho\, u_J)$ with a smooth u_J. Consider the convex mixture

$$v(\eta) \;=\; (1-\eta)\, v_G \;+\; \eta\, v_J \qquad \eta \in [0,1]\,.$$

For each η compute $C_{\min}(\rho; v(\eta))$ from $L_{\rho,G}\phi = v(\eta)$, the instantaneous power $\langle v(\eta), \mu\rangle$, the entropy production $\dot{\sigma}(\rho)$, and

$$R(\eta) \;=\; \frac{\langle v(\eta), \mu\rangle^2}{2\,C_{\min}(\rho; v(\eta))\,\dot{\sigma}(\rho)} \qquad \cos\theta(\eta) \text{ from the inner product on } \partial_x\phi \text{ and } \partial_x\mu\,.$$

Verification. Equality is expected at $\eta = 0$ where v lies on the gradient flow ray, with $R(0) \simeq 1$ and $\cos^2\theta(0) \simeq 1$. For $\eta > 0$ one expects a strict inequality with $R(\eta) = \cos^2\theta(\eta) \in [0, 1)$ that decreases as the reversible content increases.

On the same periodic discretisation with $L = 40$, $N = 512$, $\lambda = 0.10$, we observe $1 - R(0) \lesssim 2 \times 10^{-4}$ and a smooth monotone increase of the gap $1 - R(\eta)$ up to $1 - R(1) \approx 1$ when the direction is purely reversible, while $|R(\eta) - \cos^2\theta(\eta)| \lesssim 10^{-12}$ for all η.

Linear solves converge to residuals $O(10^{-12})$ throughout. This quantifies the reversible-dissipative decomposition at fixed ρ in terms of the commuting angle, and is consistent with the cost-entropy inequality and its equality case proved in the main text.

Discretisation and tolerances. Periodic domain $\mathbb{T}_L$ with representative $L = 40$, grid size $N \in \{256, 512\}$, rectangle rule for spatial integrals, pseudospectral differentiation with 2/3 de-aliasing for products, projection to the mean-zero subspace, and an SPD stabiliser $\varepsilon_{\text{mass}}$ in $L_{\rho,G}$ at the level 10^{-6}.

Linear systems are solved by preconditioned conjugate gradients to residuals $O(10^{-12})$. Reported relative discrepancies refer to these tolerances and decrease with N in the usual way.

5 Assembly and emergence

Assembly and scope. We now progress on, to assemble the symmetric and antisymmetric mobility blocks developed here with the reversible classification of the companion paper [1]. All statements are within the class of local, first order Hamiltonian theories on (ρ, S) on flat domains, with admissible boundary conditions and ρ strictly positive on its support. The two scalar certificates proved earlier in the paper drive the section: the equality certificate on the irreversible ray and the no-work certificate for the reversible cone.

Commentary. One operator sets the geometry. One inner product measures angles and lengths. The same current can flow in two perpendicular directions: down the slope (dissipation) and around the level sets (reversible motion). We make this precise and give simple tests that either pass or fail.

Standing hypotheses for this section. Unless explicitly stated otherwise, all statements below are made under the following conditions.

- *State space and positivity.* We work on a fixed spatial domain $\Omega \subset \mathbb{R}^d$ with $d \geq 1$. The density $\rho(x,t)$ is strictly positive on each connected component of its support and normalised,

$$\rho(x,t) > 0 \quad \text{for all } x \in \Omega, \qquad \int_\Omega \rho(x,t)\, dx = 1.$$

  This positivity and normalisation are exactly the hypotheses used in the functional setting of Appendix A: they guarantee that the weighted Poisson operator $L_{\rho,G}\phi := -\nabla\cdot(\rho G \nabla \phi)$ is symmetric and coercive on the mean zero subspace, so that the KKT problem $L_{\rho,G}\phi = v$ has a unique solution ϕ for each mean zero tangent v in $H^{-1}_\rho(\Omega)$.
- *Irreversible metric and pairings.* The mobility $G(x)$ is a symmetric, uniformly elliptic and bounded matrix field,

$$G(x) = G(x)^\top, \qquad \gamma_{\min} |\xi|^2 \leq \xi^\top G(x)\, \xi \leq \gamma_{\max} |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d,\ x \in \Omega,$$

  for some fixed $0 < \gamma_{\min} \leq \gamma_{\max} < \infty$. All scalar products on gradients use the weighted pairing

$$\langle a, b \rangle_{\rho,G} := \int_\Omega \rho(x)\, G(x)\, a(x)\cdot b(x)\, dx,$$

  initially defined on $H^1(\Omega)$ and then extended by density to the mean zero subspace used in the KKT solve. This is the metric singled out earlier by the cost entropy inequality and its equality cases: under these hypotheses the irreversible drift is exactly the Wasserstein gradient flow $v_{\text{irr}} = \nabla\cdot(\rho G \nabla \mu)$ for $\mu = \delta F/\delta\rho$.
- *Reversible channel and weighted Liouville condition.* The reversible structure is specified by an antisymmetric matrix field $J(x) = -J(x)^\top$ which satisfies

the weighted Liouville identity

$$\nabla_i\big(\rho(x)\,J^{ij}(x)\big) = 0$$

in the sense of distributions, that is, each column of the tensor field ρJ is divergence free with respect to Lebesgue measure. As shown in Lemma B.1, this condition is equivalent to the statement that the reversible power vanishes identically: for every sufficiently smooth test potential μ one has

$$P_{\mathrm{rev}}(\rho;\mu) := \langle v_{\mathrm{rev}},\, \mu\rangle_{H_\rho^{-1}} = 0,$$

so that the reversible component generated by J preserves the free energy $F[\rho]$ and does no work in the H_ρ^{-1} geometry.

- *Boundary classes.* The spatial domain is either periodic, or a bounded Lipschitz domain with boundary conditions chosen from the admissible classes described in Appendix A. Concretely, we impose either periodic boundaries, or no flux conditions for the irreversible and reversible fluxes,

  $$\rho G\nabla\mu\cdot n = 0, \qquad \rho J\nabla\mu\cdot n = 0 \quad \text{on } \partial\Omega,$$

  or, in the assembled (ρ, S) picture, constant S on $\partial\Omega$. These are exactly the boundary classes for which integration by parts produces no boundary contributions, so that the global mass balance and the no work identities used in the metriplectic decomposition are valid without additional boundary terms.

Commentary. Think of G as a local conductance and J as a local rotator. The first moves you downhill. The second swirls you around without changing height. We keep the rules simple and local so every identity is checkable.

5.1 Geometry, operators, and the two certificates

5.1.1 Operators and geometry

Let $F[\rho]$ be a free energy with chemical potential $\mu = \delta F/\delta\rho$. The weighted Poisson operator, its KKT potential, and the two canonical velocity components are

$$\begin{aligned} L_{\rho,G}\phi \;&:=\; -\nabla\cdot(\rho\, G\,\nabla\phi), \qquad -L_{\rho,G}\phi = v,\\ v_G \;&:=\; \nabla\cdot(\rho\, G\,\nabla\mu), \qquad v_J \;:=\; \nabla\cdot(\rho\, J\,\nabla\mu). \end{aligned}$$

It is convenient to package the symmetric and antisymmetric parts in

$$\mathcal{K} \;:=\; G + i\,J, \qquad v \;=\; \nabla\cdot(\rho\,\mathcal{K}\,\nabla\mu).$$

This notation is a mnemonic only, we *do not* assume any G-compatibility of J unless stated. All scalar identities below are proved in the real $\langle\cdot,\cdot\rangle_{\rho,G}$ geometry together with the no-work property for J. See Appendix F for the KKT characterisation of $C_{\min}$ and the mean-zero gauge conventions.

KKT solves, slice pulls, coarse graining and falsifier variants are provided in the code archive in Appendix F.

Commentary. There is one current v. Writing v with G gives the downhill part. Writing the same v with J gives the sideways part. Putting them into $\mathcal{K}$ is just a tidy way to look at both at once.

5.1.2 Scalar certificates: equality on the irreversible ray, and no-work for the reversible cone

Define the KKT potential ϕ by $-L_{\rho,G}\phi = v$. Define the control and production functionals

$$2C_{\min} := \int \rho\, G\, \nabla\phi\cdot\nabla\phi\, dx, \qquad \dot{\sigma} := \int \rho\, G\, \nabla\mu\cdot\nabla\mu\, dx.$$

Equality dial (irreversible certificate). For any admissible v,

$$\langle v, \mu\rangle^2 \le 2\, C_{\min}\, \dot{\sigma}, \qquad R := \frac{\langle v, \mu\rangle^2}{2C_{\min}\dot{\sigma}} = \cos^2\theta_{\rho,G} \in [0,1].$$

In our sign convention $v_G = -L_{\rho,G}\mu$, hence $-L_{\rho,G}\phi = v_G$ gives $\phi = \mu$ and saturates $R = 1$.

Here $\theta_{\rho,G}$ is the angle between $\nabla\phi$ and $\nabla\mu$ in the $\langle\cdot,\cdot\rangle_{\rho,G}$ metric. The proof is Cauchy-Schwarz in the weighted $H_\rho^{-1}(G)$ geometry together with the KKT characterisation of $C_{\min}$.

Equality dial and controlled rotations are exercised in Appendix F.

No-work and H_ρ^{-1} orthogonality (reversible certificate). If $J = -J^\top$ and $\nabla_i(\rho J^{ij}) = 0$ then

$$\langle v_J, \mu\rangle = \int \mu\, \nabla\cdot(\rho J \nabla\mu)\, dx = 0,$$

and, writing $v_G = \nabla\cdot(\rho G\nabla\mu)$ and $v_J = \nabla\cdot(\rho J\nabla\mu)$, the $H_\rho^{-1}(G)$ inner product of the canonical pair vanishes:

$$\langle v_G, v_J\rangle_{H_\rho^{-1}(G)} = 0,$$

since the KKT potential for v_G is $\phi = \mu$. See Appendix B for the algebraic proof of the no-work identity and the $H_\rho^{-1}(G)$ orthogonality.

Equation (5.1.2) follows by one integration by parts under the weighted Liouville identity: the boundary term vanishes, antisymmetry kills the quadratic term, and the mixed derivatives cancel.

No-work checks, anomaly detection and repair shown in Appendix F.

Commentary. Two numbers tell the whole story on a slice. The first number R is how aligned you are with the downhill slope. It reaches 1 exactly when you go straight down. The second says the sideways part does no-work at all. Together they pin down the split without ambiguity.

5.1.3 How to read the dials

Given a state ρ and a velocity v:

1. Solve $L_{\rho,G}\phi = v$ on the mean-zero subspace to obtain the KKT slope.
2. Compute R from (5.1.2). Values near 1 indicate motion along the dissipative axis, values near 0 indicate motion orthogonal to it.
3. Check the reversible certificate by measuring $\langle v_J, \mu \rangle$ and the orthogonality. Violations are linear in the size of $\nabla \cdot (\rho J)$ or in a mismatch of geometry.

These diagnostics are invariant under smooth relabellings when the operator and pairings are pulled with full Jacobian weights, and they are stable under grid refinement.

Commentary. Compute the slope that best explains v, measure its angle to the free energy slope, and check that the swirl does no-work. If any rule is broken, the numbers drop in a way that tells you exactly which rule failed.

The full dial pipeline, including KKT solve, R and $\mathcal{M}$ reporting along $v(\eta)$, is packaged with slice pulls in Appendix F.

5.2 Complex pairing, modulus, and rotation

5.2.1 Complex pairing on a slice and the guarded modulus

We introduce a complex reader that measures one and the same current in two quadratures. On a fixed state ρ define

$$\langle \nabla\phi, \nabla\psi \rangle_{\mathbb{C}} \;:=\; \int \rho\, G\, \nabla\phi \cdot \nabla\psi \, dx \;+\; i \int \rho\, \nabla\phi \cdot \mathcal{H}[\nabla\psi]\, dx,$$

where $\mathcal{H}$ is a fixed linear, skew operator defined on the same discrete subspace and grid as the KKT solve, as used in our diagnostics (a diagnostic reader for the complex pairing; it need not coincide with the system's intrinsic reversible operator J satisfying the weighted Liouville identity).

The real part is the $\langle \cdot, \cdot \rangle_{\rho,G}$ pairing that drives the equality certificate. The imaginary part captures a transverse quadrature measured with a fixed skew reader.

Define the complex modulus and the equality dial

$$\mathcal{M} \;:=\; \frac{\left|\langle \nabla\phi, \nabla\mu \rangle_{\mathbb{C}}\right|^2}{2C_{\min}\,\dot{\sigma}}, \qquad R \;:=\; \frac{\left(\Re\langle \nabla\phi, \nabla\mu \rangle_{\mathbb{C}}\right)^2}{2C_{\min}\,\dot{\sigma}} \;=\; \cos^2\theta_{\rho,G}.$$

Always

$$0 \;\le\; R \;\le\; \mathcal{M}.$$

If the imaginary quadrature is the metric Hodge rotation on the two-plane spanned by $\{\nabla\phi, \nabla\mu\}$, then $\mathcal{M} \le 1$ and equality holds. In our experiments the fixed proxy $\mathcal{H}$ saturates to numerical floor on the irreversible ray and remains within estimator tolerance along the controlled rotations described below.

The complex pairing and the associated moduli R and M are diagnostic constructs on the tangent space. They quantify alignment within the metriplectic split and are not introduced as physical observables.

Commentary. The complex pairing is a meter. Its real needle reads the downhill share. Its imaginary needle reads the sideways share. The total should not exceed one, and in our calibrated cases it sits right at one within numerical tolerance.

The complex pairing reader is evaluated and operationalised; see Appendix F.

5.2.2 Optional compatible two-plane quadrature

For readers who prefer an exact modulus, one may define a *local two-plane complex structure* J_μ at the given state by

$$\langle \nabla\mu, \nabla\mu\rangle_{\rho,G} = 1, \qquad \langle \nabla\mu, J_\mu\nabla\mu\rangle_{\rho,G} = 0, \qquad \langle J_\mu\nabla\mu, J_\mu\nabla\mu\rangle_{\rho,G} = 1,$$

and extend J_μ by rotation on the plane $\mathrm{span}\{\nabla\mu, \nabla\phi\}$, arbitrarily on its orthogonal complement. Using the induced imaginary part

$$\Im\langle \nabla\phi, \nabla\psi\rangle_{\mathbb{C}} \;:=\; \int \rho\, G\, \nabla\phi \cdot J_\mu \nabla\psi \, dx$$

yields the exact identity

$$\big|\langle \nabla\phi, \nabla\mu\rangle_{\mathbb{C}}\big|^2 \;=\; 2C_{\min}\,\dot\sigma, \qquad \text{that is} \qquad \mathcal{M} \equiv 1,$$

by construction. We do not require this construction for any theorem in the paper; it serves to clarify when the modulus is pinned to one.

Commentary. If you choose the sideways direction to be exactly a right angle to the downhill direction, then the two needles add up to a perfect circle. That pins the total to one by definition.

5.2.3 Measurement protocol for $\mathcal{M}$ and R

Given a state ρ and a velocity v:

1. Solve $L_{\rho,G}\phi = v$ on the mean-zero subspace to obtain the KKT slope.
2. Evaluate $2C_{\min}$ and $\dot\sigma$ in the ρG pairing on the same discrete subspace as the KKT solve.

3. Compute $\langle \nabla\phi, \nabla\mu \rangle_{\mathbb{C}}$ with the fixed proxy $\mathcal{H}$ on the same grid and subspace. Report R and $\mathcal{M}$ jointly.
4. For diffeomorphic pulls use full Jacobian weights in both the operator and the pairings. Report solver tolerances with grid refinement so that R approaches one on v_G and $\mathcal{M}$ remains at one within estimator floor.

> **Commentary.** Same grid, same space, same weights. Solve once, measure twice. The two numbers tell you what part is downhill and what part is sideways.

Same-grid evaluations of $2C_{\min}$, $\dot{\sigma}$ and $\langle \nabla\phi, \nabla\mu \rangle_{\mathbb{C}}$ implemented in Appendix F.

5.2.4 Rotation along the η path

Consider the controlled path $v(\eta) = (1-\eta)\, v_G + \eta\, v_J$ with $\eta \in [0,1]$. Along this path we observe that $\mathcal{M}$ remains at one within estimator floor while

$$R\big(v(\eta)\big) \;=\; \cos^2 \theta_{\rho,G}(\eta)$$

decreases monotonically from 1 to near 0 as reversible content increases. This confirms that, within the compatible two-plane geometry, the complex norm is conserved while its real share rotates into the imaginary quadrature. The path is a rotation of one current, not a splice of two models. We report the deviation $\delta_{\mathcal{M}} := |\mathcal{M} - 1|$ along $v(\eta)$ and its grid-refinement slope in Appendix F.

> **Commentary.** Turn the knob and the needle swings from downhill to sideways, but the total length stays the same. That is a clean rotation, not a switch of machines.

Controlled path $v(\eta)$ with joint reporting of $\mathcal{M}$ and R shown in Appendix F.

5.3 Relativistic assembly and slice covariance

We now show that the spatial metriplectic structure admits a natural covariant packaging on a fixed background spacetime, and that the instantaneous certificates are invariant under smooth relabellings of space. Throughout this subsection the background Lorentzian metric is purely kinematic; all dynamical structure still lives in ρ, G, J and the free energy F.

The Lorentzian formulation later is purely kinematic: the background metric g is fixed, and no relativistic dynamics are claimed beyond the tensor transformation rules used.

5.3.1 Complex four-current and conservation

Let $(\mathcal{M}, g)$ be a fixed time oriented globally hyperbolic spacetime with Levi-Civita connection ∇ and background volume form dV_g. We treat ρ as a strictly positive scalar density with respect to dV_g so that $\rho\, dV_g$ is the physical mass measure, with

$$\int_{\Sigma_t} \rho\, d\Sigma_t \;=\; 1$$

for each Cauchy slice Σ_t. Indices are raised and lowered with $g_{\mu\nu}$.

We take $G^{\mu\nu}$ and $J^{\mu\nu}$ to be smooth tensor fields with the following properties.

- $G^{\mu\nu} = G^{\nu\mu}$ is symmetric, uniformly bounded, and positive on spatial directions: for every future pointing unit normal n_μ to each Σ_t, one has $G^{\mu\nu} n_\nu = 0$ and there exist constants $0 < \gamma_{\min} \leq \gamma_{\max} < \infty$ such that

  $$\gamma_{\min}\, h_{ij}\, \xi^i \xi^j \;\leq\; G^{ij}\, \xi_i \xi_j \;\leq\; \gamma_{\max}\, h_{ij}\, \xi^i \xi^j$$

  for all spatial covectors ξ, where h_{ij} is the induced Riemannian metric on Σ_t.
- $J^{\mu\nu} = -J^{\nu\mu}$ is antisymmetric and purely spatial in the same sense, with the weighted Liouville condition

  $$\nabla_\mu\big(\rho\, J^{\mu\nu}\big) \;=\; 0$$

  interpreted in the distributional sense. Equivalently, each column of the tensor density $\rho J^{\mu\nu}$ is divergence free with respect to the Lebesgue measure induced by g.

Given a smooth chemical potential $\mu\colon \mathcal{M} \to \mathbb{R}$ we define the complex four-current

$$j^\mu \;:=\; -\,\rho\, \mathcal{K}^{\mu\nu}\, \nabla_\nu \mu, \qquad \mathcal{K}^{\mu\nu} \;:=\; G^{\mu\nu} + i\, J^{\mu\nu}.$$

We write

$$\Re j^\mu = -\,\rho\, G^{\mu\nu} \nabla_\nu \mu, \qquad \Im j^\mu = -\,\rho\, J^{\mu\nu} \nabla_\nu \mu.$$

Lemma 5.1 (*Conservation of the reversible four-current*). *Under the hypotheses above, for every smooth μ one has*

$$\nabla_\mu(\Im j^\mu) \;=\; 0.$$

Proof. Compute

$$\nabla_\mu(\Im j^\mu) = \nabla_\mu\big(\rho\, J^{\mu\nu}\, \nabla_\nu \mu\big) = \big(\nabla_\mu(\rho J^{\mu\nu})\big)\, \nabla_\nu \mu + \rho\, J^{\mu\nu}\, \nabla_\mu \nabla_\nu \mu.$$

The first term vanishes by the weighted Liouville identity (5.3.1). Since ∇ is torsion free the Hessian $\nabla_\mu \nabla_\nu \mu$ is symmetric under exchange of μ and ν, whereas $J^{\mu\nu}$ is antisymmetric, so

$$J^{\mu\nu}\, \nabla_\mu \nabla_\nu \mu = -\, J^{\nu\mu}\, \nabla_\mu \nabla_\nu \mu = -\, J^{\mu\nu}\, \nabla_\mu \nabla_\nu \mu$$

and hence $J^{\mu\nu}\, \nabla_\mu \nabla_\nu \mu = 0$. The second term therefore vanishes and the result follows. □

The real part encodes the dissipative divergence. In general $\nabla_\mu(\Re j^\mu)$ does not vanish. On each time slice Σ_t with unit normal n_μ we decompose $\Re j^\mu$ into normal and tangential parts

$$\Re j^\mu = (j^\perp)\, n^\mu + j^\mu_\parallel, \qquad n_\mu j^\mu_\parallel = 0,$$

and define the spatial irreversible flux by $v^i := j^i_\parallel$ in adapted coordinates. A short calculation using the Gauss formula and the fact that $G^{\mu\nu}$ is purely spatial shows that the spatial divergence of $\Re j^\mu$ on Σ_t is exactly the weighted divergence generated by the operator

$$L_{\rho,G}\phi \;=\; -\nabla_i\big(\rho\, G^{ij}\, \nabla_j\phi\big)$$

studied in Sections 5.1.1 and 5.2.1. In particular, writing $\mu_t = \delta F/\delta\rho$ on each slice, the metriplectic irreversible evolution

$$\partial_t\rho = \nabla_i\big(\rho\, G^{ij}\, \nabla_j\mu_t\big) = -\,L_{\rho,G}\mu_t$$

can be rewritten as the local conservation law

$$\partial_t\rho + \nabla_i(\Re j^i) = 0, \qquad \Re j^i = -\,\rho\, G^{ij}\, \nabla_j\mu_t,$$

for the real part of the four-current (5.3.1), , while the reversible part is encoded in the conserved imaginary current of Lemma 5.1.

Finally, the complex pairing used for the equality certificates can be lifted to spacetime by

$$\langle\nabla\phi, \nabla\mu\rangle_{\rho,\mathcal{K}} := \int_{\Sigma_t} \rho\, \mathcal{K}^{ij}\, \nabla_i\phi\, \overline{\nabla_j\mu}\, d\Sigma_t,$$

which reduces on each slice to the spatial complex pairing already used in the definition of the equality dial R and the modulus $\mathcal{M}$. The same operator $L_{\rho,G}$ and the same pairing therefore arise as spatial sections of a single complex four-current j^μ.

Commentary. In spacetime language there is one complex current. Its real part gives the downhill irreversible flux and enters the continuity equation, while its imaginary part is a conserved sideways current fixed by the weighted Liouville condition. Taking a time slice recovers exactly the spatial operators and pairings already used for the equality and no-work certificates.

5.3.2 Slice covariance under smooth relabellings

We now formalise the statement that the instantaneous certificates depend only on the geometry induced by (ρ, G, J) and not on a particular coordinate chart on a spatial slice.

Let Σ be a fixed time slice with local coordinates x, and let $y = y(x)$ be a smooth diffeomorphism of Σ with Jacobian matrix $Dy(x)$ and determinant $\mathrm{Jac}_y(x) := \det Dy(x)$. We define the pushed fields on the y chart by the standard tensorial rules

$$\rho'(y) := \frac{\rho(x)}{\mathrm{Jac}_y(x)}, \qquad G'^{ij}(y) := \frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial x^\ell} G^{k\ell}(x), \qquad J'^{ij}(y) := \frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial x^\ell} J^{k\ell}(x),$$

with the scalar potential μ pulled by composition, $\mu'(y) := \mu(x)$. The spatial volume form transforms as $dx = \mathrm{Jac}_y^{-1}(x)\, dy$, so that $\rho\, dx = \rho'\, dy$.

Proposition 5.2 (*Slice covariance of the certificates*). *Under the change of variables* (5.3.2) *the spatial operator $L_{\rho,G}$, the real pairing $\langle\cdot,\cdot\rangle_{\rho,G}$, and the complex pairing $\langle\cdot,\cdot\rangle_{\rho,\mathcal{K}}$ are invariant in the following sense.*

(a) For every mean-zero test function ϕ one has

$$L_{\rho,G}\phi(x) = L_{\rho',G'}\phi'(y)\big|_{y=y(x)}, \qquad \phi'(y) := \phi(x).$$

(b) For every pair of gradients $\nabla\phi$, $\nabla\mu$ on Σ one has

$$\int_\Sigma \rho\, G\, \nabla\phi\cdot\nabla\mu\, dx = \int_\Sigma \rho'\, G'\, \nabla_y\phi'\cdot\nabla_y\mu'\, dy,$$

and similarly for the complex pairing with $\mathcal{K} = G + iJ$.

Consequently the equality dial R and the modulus $\mathcal{M}$, which are constructed from these pairings and from the KKT potentials associated with $L_{\rho,G}$, are invariant under smooth relabellings of the spatial slice.

Proof. The transformation rules (5.3.2) are exactly those of a scalar density ρ and rank two contravariant tensors G^{ij}, J^{ij}. A direct change of variables gives

$$\int_\Sigma \rho\, G^{ij}\, \nabla_i\phi\, \nabla_j\mu\, dx = \int_\Sigma \rho'(y)\, G'^{ij}(y)\, \nabla_{y_i}\phi'(y)\, \nabla_{y_j}\mu'(y)\, dy,$$

since the Jacobian factors from dx and from ρ' and G' cancel exactly. The same holds for the complex pairing with $\mathcal{K}$.

For the operator, recall that $L_{\rho,G}\phi = -\nabla_i(\rho\, G^{ij}\nabla_j\phi)$ is defined as the divergence of the flux density $\rho\, G\nabla\phi$. Because both ρ and G transform as above, and because the covariant divergence of a vector density is intrinsic, one finds $L_{\rho,G}\phi = (L_{\rho',G'}\phi')\circ y$. The construction of the KKT potential for a given tangent v only uses $L_{\rho,G}$ and the pairing $\langle\cdot,\cdot\rangle_{\rho,G}$ on the mean-zero subspace, so the potential and all quadratic forms built from it are invariant under the relabelling. The equality dial R and the modulus $\mathcal{M}$ are ratios of such quadratic forms and are therefore invariant. □

Practical note. In all reported numerical tests we discretise both the original and the relabelled slice on the same uniform grid, perform the KKT solve and all real and complex pairings on the corresponding mean-zero subspace with respect to the pushed measure $\rho'\, dy$, and apply identical finite difference stencils and spectral de-aliasing. Under these conditions the equality dial R and the modulus $\mathcal{M}$ agree between the two charts to numerical floor, in line with Proposition 5.2.

Commentary. You can smoothly relabel space, push forward ρ, G, J and the potentials with the correct Jacobian weights, and recompute everything. The operator $L_{\rho,G}$, the complex current j^μ and the instantaneous dials R and $\mathcal{M}$ do not care about the chosen coordinates, only about the underlying geometry encoded by (ρ, G, J).

5.4 Linear response and loop phase

5.4.1 Linear response, causality, and Kramers-Kronig

Linear response is computed from the resolvent of the linearised generator on a fixed Fourier mode k. The resolvent is taken on the $H^{-1}_\rho(G)$ tangent with energy pairing $\langle\cdot,\cdot\rangle_{\rho,G}$. For a harmonic probe at frequency ω,

$$(i\omega I - \mathcal{L}_k)\,\widehat{\delta\rho} \;=\; \widehat{B}\,\widehat{\mu}, \qquad \chi(\omega,k) \;:=\; \langle\nabla\phi, \nabla\mu\rangle_{\mathbb{C}},$$

where $-L_{\rho,G}\phi = \delta v$ is the KKT solve for the induced velocity δv and the complex pairing uses the same discretisation and weights as in Section B.

Causality and analyticity guard. Assume the resolvent exists and the response is causal and stable so that $\chi(\omega,k)$ is analytic in the upper half-plane $\{\Im\omega > 0\}$. Then the Kramers-Kronig relations link the two quadratures in frequency:

$$\Re\chi(\omega,k) \;=\; \frac{1}{\pi}\,\text{p.v.}\!\int_{-\infty}^{\infty} \frac{\Im\chi(\omega',k)}{\omega'-\omega}\,d\omega', \qquad \Im\chi(\omega,k) \;=\; -\frac{1}{\pi}\,\text{p.v.}\!\int_{-\infty}^{\infty} \frac{\Re\chi(\omega',k)}{\omega'-\omega}\,d\omega'.$$

Operationally, a single scale factor $\kappa_{\mathcal{H}}$ is applied once per grid to account for the discrete estimator used in $\mathcal{H}$.

Measurement protocol. Fix k, choose a grid of frequencies $\{\omega_j\}$. For each ω_j solve one KKT system to obtain ϕ, then evaluate $\chi(\omega_j,k) = \langle\nabla\phi,\nabla\mu\rangle_{\mathbb{C}}$ on the same subspace and report $\Re\chi, \Im\chi$. Compare $\Im\chi$ against the Hilbert transform of $\Re\chi$ with a single calibration factor.

Commentary. Shake the system at a chosen pitch and watch two needles: in-phase and out-of-phase. If the response is causal, the two are tied together by a standard integral link. We compute both with the same ruler so they can be trusted.

Frequency sweeps, resolvent solves and KK comparison in Appendix F.

5.4.2 Geometric phase on slow parameter loops

Let $\lambda = (\lambda^1, \lambda^2)$ denote two slow external controls, for example background amplitude and reversible phase. Along a quasi-static loop $\lambda(s)$, $s \in [0,1]$, define the loop phase of the reader by

$$\Phi[\lambda] \ := \ \text{unwrap arg}\left(\langle \nabla\phi(\lambda), \nabla\mu(\lambda) \rangle_{\mathbb{C}} \right).$$

Empirically, Φ accumulates a small but nonzero total that flips sign under loop reversal and scales with enclosed area for small loops. The phase is invariant under $\mu \mapsto \mu + \text{const}$ and shifts by a boundary term under $\mu \mapsto \mu + \partial_s \Lambda(\lambda)$.

Operational recipe. Trace a small rectangle in (λ^1, λ^2)-space, sample the complex pairing at the four corners and along the edges, unwrap the phase, and compare the total for opposite orientations.

> **Commentary.** Change two knobs in a loop. When you come back, a tiny extra angle remains. Reverse the loop and the sign flips. This is the simplest fingerprint of underlying curvature.

Loop phase and winding measurements assessed in Appendix F.

5.5 Holonomy, coarse graining, and path cost

We now assemble three simple diagnostics that use the complex reader beyond a single slice: holonomy on a two parameter control family, its behaviour under coarse graining, and the path cost associated with the metric generated by $L_{\rho,G}$. All three are implemented with the same KKT machinery and pairings as in the previous subsections.

5.5.1 Holonomy of the complex reader on a control rectangle

Let $\lambda^1 = a$ be the amplitude of a background density modulation and $\lambda^2 = \theta$ a mixing angle in a two plane spanned by irreversible and reversible velocities. On each point $\lambda = (a, \theta)$ we fix a smooth state ρ_λ and compute the corresponding chemical potential μ_λ, induced current v_λ and KKT slope ϕ_λ as before. The complex reader

$$Z(\lambda) \ := \ \langle \nabla\phi_\lambda, \nabla\mu_\lambda \rangle_{\mathbb{C}}$$

is then evaluated on a uniform grid in (a, θ).

On this grid we define a discrete Berry curvature by summing phase increments of Z around each elementary plaquette,

$$F_{mn} \ := \ \Delta \arg Z_{mn}$$

with the usual branch choice for the complex logarithm, and a Chern number estimator

$$C \ := \ \frac{1}{2\pi} \sum_{m,n} F_{mn}.$$

In the smooth test family used here the reader is nonzero on the entire control rectangle and $|Z|$ remains bounded away from zero. Numerically we observe a smooth curvature map with total flux consistent with zero and a rounded Chern index

$$C \approx 0,$$

together with small KKT residuals and exact weighted Liouville diagnostics. In other words, the assembled metriplectic structure lives in a topologically trivial sector for these controls.

Holonomy maps, curvature plots and Chern estimates are reported in `17_holonomy_curvature_map.py`.

> **Commentary.** We treat amplitude and mixing angle as slow knobs, read the complex pairing on a grid, and look at the phase around each little square. In the family studied here the phase is smooth, the total curvature cancels, and the Chern count comes out as zero within numerical floor.

5.5.2 Coarse graining invariance of phase and curvature

To test robustness under coarse graining we construct a Gaussian smoothed density $\rho^{(\ell)}$ and corresponding irreversible velocity at a fixed coarse graining length ℓ, using the same background modulation and reversible two plane. On the same control grid we recompute the reader

$$Z^{(\ell)}(\lambda) \;:=\; \langle \nabla\phi_\lambda^{(\ell)}, \nabla\mu_\lambda^{(\ell)} \rangle_{\mathbb{C}}$$

with the KKT solve and pairings carried out on the coarse state.

We compare fine and coarse results by the phase difference $\Delta \arg Z$ and the modulus ratio $|Z^{(\ell)}|/|Z|$ over the grid, together with the Berry curvature and Chern estimates for both. In the regime tested here we find:

- The phase difference is small, with root mean square at the level of 10^{-4} radians.
- The Chern estimates for fine and coarse grids agree to numerical floor and remain at $C \approx 0$.
- The modulus ratio $|Z^{(\ell)}|/|Z|$ is close to a constant over control space, reflecting a nearly uniform rescaling of the reader by coarse graining.

This supports a simple picture: in the class studied here, coarse graining modifies the overall strength of the complex reader while leaving its phase and associated holonomy invariant within numerical tolerance. The topological content, here trivial, behaves as an effective renormalisation group invariant.

Fine and coarse comparisons and curvature differences are implemented in `19_holonomy_coarsegrain_invariance.py`.

> **Commentary.** We blur the state and recompute everything. The overall size of the complex reading changes by a nearly constant factor, but the phase picture and the total curvature stay the same. The large scale geometry is therefore insensitive to this level of smoothing.

5.5.3 Synthetic monopole testbench

To check that the holonomy machinery is sensitive to genuine defects and not only to smooth trivial configurations, we couple it to a synthetic complex field on a control space that is independent of the metriplectic dynamics. On a rectangular grid $(\lambda^1, \lambda^2) \in \mathbb{R}^2$ we prescribe

$$Z_{\mathrm{syn}}(\lambda^1, \lambda^2) \;:=\; \exp\left(i \arg(\lambda^1 + i\lambda^2)\right) \mathrm{e}^{-\alpha(\lambda_1^2+\lambda_2^2)}, \qquad \alpha > 0,$$

which has a simple phase vortex at the origin and no singularities away from zero.

Applying exactly the same plaquette based curvature estimator and loop winding measurements as above, we recover

$$\sum F \approx 2\pi, \qquad C \approx 1,$$

for the total curvature, and loop windings $n \approx 1$ for loops encircling the origin and $n \approx 0$ for loops that do not. This provides an explicit calibration of the discrete Berry curvature and Chern estimators and a basic check that the code path used on the metriplectic reader responds correctly when a single controlled defect is present.

The synthetic testbench is provided in `22_holonomy_synthetic_monopole.py`.

Commentary. We feed the same machinery a toy field with a known phase vortex. The curvature sum comes out as one quantum, and loops around the origin pick up one turn while distant loops do not. This reassures us that the trivial Chern result in the metriplectic test family is a property of that family, not a blind spot of the tool.

5.5.4 Path cost and entropy change

Finally we connect the equality dial and complex reader back to the metric induced by $L_{\rho,G}$. For a time dependent protocol ρ_t with induced instantaneous velocity v_t we define the instantaneous cost in the weighted $H_\rho^{-1}(G)$ geometry by

$$C(t) \;:=\; \frac{1}{2}\int \rho_t\, G\, \nabla\phi_t \cdot \nabla\phi_t \, dx, \qquad -L_{\rho_t,G}\phi_t = v_t,$$

and from this the path action and path length

$$A[\rho_\cdot] \;:=\; \int_0^T C(t)\, dt, \qquad \mathcal{L}[\rho_\cdot] \;:=\; \int_0^T \sqrt{C(t)}\, dt.$$

We compare two simple protocols that interpolate between the same initial and final amplitudes a_{start} and a_{end} over the same time T: a linear schedule and a gently wiggled schedule which adds an oscillatory component. In both cases the Shannon entropy change ΔS is the same within numerical tolerance, but the metric quantities differ by order one factors. In the representative runs reported here

$$\frac{A_{\text{wiggle}}}{A_{\text{linear}}} \approx 4, \qquad \frac{\mathcal{L}_{\text{wiggle}}}{\mathcal{L}_{\text{linear}}} \approx 1.7,$$

with KKT residuals at solver tolerance.

This confirms that the $H_\rho^{-1}(G)$ metric induced by $L_{\rho,G}$ defines a genuine path dependent cost that is not determined by the entropy change alone. The present paper does not attempt a full optimality or speed limit theory, but these simple protocols already show that within the same closure the geometry picks out preferred paths at fixed endpoints and time. In this sense $C(t)$ and the induced action $A[\rho.]$ play the role of a thermodynamic length functional in the $H_\rho^{-1}(G)$ geometry, in parallel with the control space metrics studied in [29–31].

Protocol generation, cost evaluation and entropy tracking are implemented in `20_protocol_cost_vs_entropy.py`.

Commentary. We move between the same start and end states, in the same time in two different ways. The entropy drop is the same, but the metric cost is not. Straight paths are cheaper than wiggled ones in the geometry fixed earlier, which is exactly what a meaningful distance measure should report.

5.6 Minimal slice projections

5.6.1 Electrodynamic-like projection on a slice (Analogy)

In two dimensions write $J^{ij}(x) = c(x)\,\epsilon^{ij}$ and define $u_J := J\nabla\mu = c\,R_{90^\circ}\nabla\mu$, $j_J := \rho u_J$, $E := -\nabla\mu$, and $B := c\,\rho$. The continuity equation $\partial_t\rho + \nabla\cdot(\rho u_J) = 0$ gives

$$\partial_t B + \nabla\cdot(B\,u_J) \;=\; \rho\,\partial_t c \;+\; \rho\,u_J\cdot\nabla c,$$

which vanishes only when c is constant in space and time. The Faraday-style expression $-\nabla\cdot(R_{90^\circ}(B\,E))$ is an analogy that relates structures on a slice and is not asserted as an exact identity here.

Weighted Liouville and constant matrices. A spatially constant matrix J_0 satisfies $\nabla_i(\rho J_0^{ij}) = 0$ only when ρ is constant. For variable ρ, a sufficient Liouville-compatible choice is $J(x) = c\,\rho(x)^{-1}\epsilon$ with constant c, for which $\nabla_i(\rho J^{ij}) = \nabla_i(c\,\epsilon^{ij}) = 0$.

Anomaly falsifier. If the weighted Liouville law is violated, $\nabla_i(\rho J^{ij}) = s^j \neq 0$, the no-work identity acquires a source term

$$\dot\sigma_{\text{anom}} \;=\; \int s^j\,\partial_j\mu\,dx,$$

which is detected as a linear rise in the reversible power. Restoring $\nabla \cdot (\rho J) = 0$ cancels the anomaly.

Slice projection, reversible power readouts and Maxwell-style checks in Appendix F.

Scope of the analogy. This projection is a bookkeeping device on a slice. It does not assert propagating density waves in the reversible sector within the present closure. Wave-like behaviour requires extra structure beyond the scalar density channel.

> **Commentary.** On a flat sheet, E points downhill, B stores how strong the swirl is, and the identity above looks like Faraday's law with clear source terms when you change the swirl strength. If you break the swirl rule, the no-work meter lights up.

Anomaly injection and compensator construction implemented in Appendix F.

5.6.2 Optical-metric projection (Analogy)

Identify $L_{\rho,G}$ with the Laplace-Beltrami operator for the optical metric $g_{ij} \propto (\rho G)^{-1}$. Two limits are useful.

Irreversible WKB optics. In the short-scale irreversible regime, packet centres follow geodesics of g and focus where G increases. Quasistatic problems reduce to the Poisson-type law

$$\nabla \cdot (\rho G \nabla \mu) = \sigma,$$

with σ an imposed source.

Reversible rays belong to the companion paper. Hamiltonian ray dynamics for reversible packets and their phase transport are part of the linear Schrödinger sector in the companion paper. We keep the reversible slice here as incompressible transport without new ray claims.

Lensing analogy. Spatial gradients of ρG bend irreversible rays with deflection angles consistent with optical lensing under the correspondence $n^2 \sim 1 + 2\Phi/c^2$ and $n = (\rho G)^{-1/2}$. This is an analogy, not an extra claim about the base dynamics.

> **Commentary.** Treat $(\rho G)^{-1}$ as an index of refraction. Higher index bends paths more. That gives a clean picture for the downhill flow and a simple Poisson law for slow problems, while reversible ray stories live next door in the companion paper.

Optical metric Poisson problems and lensing-style deflection in Appendix F.

5.7 Covariant invariants, anomaly inflow, and quantised holonomy

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5.7.1 Covariant form of the instantaneous invariants

Let $(\mathcal{M}, g_{\mu\nu})$ be a fixed background spacetime and let Σ be a spacelike hypersurface with unit normal n_μ and induced volume element $d\Sigma$. For tangent vectors on Σ choose a unit vector t^μ orthogonal to n_μ in the measurement direction. We package the slice pairing used for R and $\mathcal{M}$ into the covariant form

$$\langle \nabla\phi, \nabla\mu \rangle_{\mathbb{C}}^{(\Sigma)} = \int_\Sigma \rho \left(n_\mu G^{\mu\nu} \nabla_\nu \phi \, n_\lambda \nabla^\lambda \mu + i \, n_\mu J^{\mu\nu} \nabla_\nu \phi \, t_\lambda \nabla^\lambda \mu \right) d\Sigma.$$

Its real part is the dissipative quadratic form that appears in the equality certificate on the slice, while the imaginary part probes a transverse reversible quadrature in the t^μ direction.

Under the weighted Liouville condition

$$\nabla_\mu (\rho J^{\mu\nu}) = 0,$$

the reversible contribution defines a conserved current, in the sense that changing the slice Σ by equal-time re-slicing or mild boosts that preserve the measurement weights leaves the imaginary part of the pairing unchanged. In particular, when the same (ρ, G, J) data are pulled to a new slice with the usual tensor rules, the dial R and the modulus $\mathcal{M}$ constructed from $\langle \nabla\phi, \nabla\mu \rangle_{\mathbb{C}}^{(\Sigma)}$ agree between slices up to solver tolerance in the configurations we test.

Operationally, we implement these transformations by pulling (ρ, G, J) and the scalar potentials to boosted or tilted slices, recomputing the KKT potential on the corresponding mean-zero subspace, and re-evaluating the pairings with the pushed measure. Within numerical floor the reported values of R and $\mathcal{M}$ remain unchanged, in line with the covariant picture developed earlier for the complex four-current.

Equal-time re-slicing and boost covariance checks are documented in Appendix F, see in particular `12_covariance_boost.py`.

> **Commentary.** We can tilt the cutting plane in spacetime, push forward the data with the correct tensor rules, and recompute the meters. The two numbers stay put within numerical floor as long as the weighted swirl rule holds, which is exactly what one expects from a conserved current measured in a fixed geometry.

5.7.2 Anomaly inflow and cancellation

When the weighted Liouville law is violated,

$$\nabla_i (\rho J^{ij}) \;=\; s^j \neq 0,$$

the reversible no-work identity acquires a source

$$\dot{\sigma}_{\text{anom}} \;=\; \int s^j \,\partial_j \mu \, dx.$$

This term is interpreted as an inflow from an auxiliary boundary current. If there exists $J'^{ij} = -J'^{ji}$ such that

$$\nabla_i \big(\rho (J^{ij} + J'^{ij})\big) \;=\; 0,$$

then the anomaly cancels and the no-work certificate is restored. This provides an explicit repair mechanism and a practical diagnostic for violation.

Commentary. If the swirl rule is broken, power leaks in. Add a compensating swirl so the weighted rule holds again and the leak stops.

5.7.3 Holonomy, winding, and where quantisation appears

Let $\Gamma \subset \mathbb{R}^2$ be a loop in a two parameter control space, for example amplitude and reversible mixing angle, and consider the complex reader

$$Z(\lambda) \;=\; \langle \nabla\phi_\lambda, \nabla\mu_\lambda \rangle_{\mathbb{C}}$$

evaluated along $\lambda \in \Gamma$. The loop phase is defined by the unwrapped argument

$$\Phi[\Gamma] \;:=\; \text{unwrap arg}\, Z(\lambda),$$

and the associated winding number

$$n[\Gamma] \;=\; \frac{1}{2\pi} \oint_\Gamma d \,\arg Z(\lambda)$$

is integer valued whenever Z is smooth and nonzero on Γ and has only isolated zeros in the enclosed region.

In the smooth metriplectic test family studied here the reader remains nonvanishing on the control rectangles we consider, the Berry curvature integrates to zero, and discrete Chern estimates give $C \approx 0$. Loops placed inside this region therefore have zero winding. This is consistent with the absence of defects in the chosen control class rather than a limitation of the holonomy construction.

To see quantised holonomy explicitly with the same code path, we couple the plaquette and loop machinery to a synthetic complex field on control space with a single phase vortex. In that setting the total curvature sum yields $C \approx 1$ and loops that encircle the vortex have $n \approx 1$, while far loops have $n \approx 0$. This serves as a calibrated example of quantised holonomy within the present framework.

Control space curvature maps, coarse graining stability, loop phases and the synthetic monopole testbench are documented in `17_holonomy_curvature_map.py`, `19_holonomy_coarsegrain_invariance.py` and `22_holonomy_synthetic_monopole.py` in Appendix F.

Commentary. In the metriplectic test family we actually use, the complex reading stays smooth and never vanishes on the control rectangles, so the net winding comes out as zero. The same machinery sees a single quantum of curvature and unit winding as soon as we feed it a simple vortex. Quantisation shows up when defects are present, and the trivial case stays trivial.

5.8 Sectoriality and ultraviolet control

Linearising the dissipative generator about a smooth background with $\rho > 0$ and uniformly elliptic G gives a sectorial operator with real part bounded above by a negative quadratic symbol. In Fourier variables,

$$\Re\,\omega(k) \;=\; -\,D_2\,|k|^2 \;-\; D_4\,|k|^4 \;+\; \mathcal{O}(|k|^6),$$

with $D_2 > 0$ determined by the local coefficients and boundary class. The reversible part is skew with respect to the energy pairing and does not change the spectral abscissa. Consequently the resolvent admits sectorial bounds and energy decays monotonically.

Composite quadratic observables built from the susceptibility $\chi(k)$ decay at least as $|k|^{-2}$ in the ultraviolet under the same hypotheses. See Appendix H for Fourier-mode oracles and decay fits that illustrate these bounds.

Commentary. High frequencies are tamed by diffusion. The swirl does not undo that because it is a perfect sideways motion. The standard energy norms stay finite without any ad hoc fixes.

Scripts: For spectral fits and sectorial resolvent bounds and UV decay tests for composite observables see Appendix F.

5.9 The emergent picture

One operator and one pairing fix the local geometry. Within that geometry a single current is decomposed into two quadratures that are both directly measurable. The dissipative sector carries an equality certificate that saturates on the irreversible ray and a curvature coercivity bound on the $H^{-1}_{\rho}(G)$ unit sphere. The reversible sector carries a no-work certificate and an orthogonality statement in the same metric, encoded by antisymmetry and the weighted Liouville identity. The complex reader ties the two into a single dial and a guarded modulus on a compatible two-plane, without introducing any new dynamical degrees of freedom.

The relativistic packaging is a kinematic lift of these slice statements: the same operator $L_{\rho,G}$ and the same pairings appear as spatial sections of a complex four-current whose imaginary part is conserved under the weighted Liouville rule. Holonomy, loop phase and slice covariance all reduce to re-evaluating the same dials after pushing (ρ, G, J) to a new slice with the correct tensor weights.

The role of the code archive is to certify this picture in a way that is accessible to dynamics and quant-ph audiences. Each diagnostic is designed as a small, local falsifier: equality on the irreversible ray, no-work in the reversible cone, slice and boost covariance, coarse-graining commutators, holonomy and sectoriality. When an axiom is relaxed the corresponding meter moves in a controlled way; when all axioms hold the meters lock in and stay locked under refinement.

Commentary. At the end the picture is simple: one geometry, one current, two clean readings. The same metres are used everywhere, and the failure modes are mapped. This is the level at which the reversible-dissipative split becomes directly testable rather than schematic.

5.9.1 Validations

1. **Equality dial.** On $v = v_G$ the dial R saturates to one under grid refinement and tighter solver tolerances. See `03_entropy_phase_eta.py` in the code archive in Appendix F.
2. **Conserved modulus.** With the fixed proxy $\mathcal{H}$, the modulus $\mathcal{M}$ remains pinned at one within estimator floor along the controlled rotation $v(\eta)$, with R decreasing monotonically. Compatible two-plane construction gives $\mathcal{M} \equiv 1$ by definition. See `03_entropy_phase_eta.py`.
3. **Kramers-Kronig response.** Under the analyticity guard the measured $\Im\chi(\omega, k)$ matches the discrete Hilbert transform of $\Re\chi(\omega, k)$ up to a single calibration constant for the estimator. See `02_kk_resolvent.py`.
4. **Slice covariance.** Under smooth relabellings with full Jacobian weights in both operator and pairings, R and $\mathcal{M}$ are invariant to numerical floor on matched subspaces. See `04_diffeo_slice.py`.
5. **Boost and equal-time re-slicing.** Equal-time re-slicing and boost covariance preserve R and $\mathcal{M}$ within solver tolerance when weights and subspaces are matched. See `12_covariance_boost.py`.
6. **Coarse graining.** Gaussian coarse-graining commutator defect scales as ℓ^2 across a decade in ℓ with smooth filters. See `05_coarsegrain_commutator.py`.
7. **Sectoriality.** The dissipative spectrum fits $\Re\omega(k) \approx -D_2|k|^2 - D_4|k|^4$ with $D_2 > 0$ over admissible backgrounds. See `08_sectoriality_scan.py`.
8. **Ultraviolet decay.** Composite observables built from $\chi(k)$ obey at least $|k|^{-2}$ decay in the ultraviolet and require no external renormalisation within the stated class. See `16_uv_sectoriality.py`.
9. **Loop phase and winding.** Slow parameter loops exhibit a geometric phase that flips sign under loop reversal and scales with enclosed area for small loops. Winding number is integer stable away from branch points. See `07_holonomy_loop.py` and `14_holonomy_quantisation.py`.
10. **Slice projections.** Electrodynamic and optical-metric projections reproduce the advertised readouts on slices with the stated caveats. See `09_em_slice_2d.py`, `11_maxwell_slice.py`, `10_optical_metric_poisson.py`, `15_optical_gravity_lensing.py`.

5.9.2 Falsifiers

1. **Pulling G outside divergence.** Replacing $-\nabla \cdot (\rho G \nabla \cdot)$ by $-\rho \nabla \cdot (G \nabla \cdot)$ collapses R. The gap $1-R$ grows with mobility contrast and grid refinement. See `06_structure_falsifier.py`.
2. **Wrong tangent metric.** Using a pairing not induced by ρG lowers both R and $\mathcal{M}$ in proportion to the mismatch, even when the operator is correct. See `06_structure_falsifier.py`.
3. **Nonsmooth filters.** Box filters break the ℓ^2 commutator law and introduce spurious plateaus in the dial. See `05_coarsegrain_commutator.py`.
4. **Liouville violation.** If $\nabla \cdot (\rho J) \neq 0$ then the reversible no-work identity fails with a linear anomaly $\dot{\sigma}_{\text{anom}}$. See `09_em_slice_2d.py`.
5. **Anomaly inflow and repair.** Introducing a compensator J' that restores $\nabla \cdot \big(\rho(J + J')\big) = 0$ cancels the anomaly and restores the certificate. See `13_anomaly_inflow.py`.
6. **Incompatible quadrature.** A fixed imaginary reader not compatible with the local two-plane structure lowers $\mathcal{M}$ away from one even along $v(\eta)$. See `03_entropy_phase_eta.py`.

> **Commentary.** The checks are minimal, targeted and diagnostic. When a rule is broken the meters move in a predictable way. When all rules hold the meters lock in.

5.10 Limits and frontier map

Ellipticity and positivity. All identities are proved under $\rho > 0$ and uniformly elliptic G. As either condition weakens, conditioning degrades and numerical fronts appear. Trends in R and $\mathcal{M}$ remain monotone up to the breakdown threshold. See `08_sectoriality_scan.py` for spectral early warning and `16_uv_sectoriality.py` for UV behaviour.

Metric fidelity. The $H_{\rho}^{-1}(G)$ geometry is essential. Mismatched metrics decouple information distance from free energy curvature and break the equality dial in controlled ways. See `06_structure_falsifier.py`.

Boundary classes. The stated boundary classes remove boundary terms. Other classes may require correctors and are out of scope here. Slice projections document boundary effects explicitly. See `11_maxwell_slice.py`.

> **Commentary.** The rules live inside a safe box. Near the edges the numbers get noisy first, then fail. We map where that happens and how it shows up in the meters.

5.11 Minimal reproducibility kit

Domain and discretisation. Periodic domain, Fourier de-aliasing at the two thirds rule, mean-zero KKT solve, pairings on the same discrete subspace, and full Jacobian weights for diffeomorphic pulls. See `03_entropy_phase_eta.py` and `04_diffeo_slice.py`.

Solver tolerances. Report linear solver residuals and grid refinement. In our runs R on v_G pins to one within 10^{-6} to 10^{-8} and $\mathcal{M}$ remains at one within estimator floor along $v(\eta)$. All numbers are reported with identical stencils and weights.

Response and phase tools. Resolvent based $\chi(\omega, k)$ with one KKT solve per frequency and KK comparison with a single calibration factor for the discrete Hilbert transform. Loop phase reads the unwrapped argument of $\langle \nabla\phi, \nabla\mu \rangle_{\mathbb{C}}$ and its sign flip under loop reversal. See `02_kk_resolvent.py` and `07_holonomy_loop.py`.

6 Fisher scalar sector, illustration and interpretation

Having assembled the symmetric and antisymmetric mobility blocks in 5 into a single dial-and-modulus picture over the (ρ, S) hydrodynamics, we now record a scalar sector built from the same density, the same weighted operator $L_{\rho,G}$ and the same Fisher quadratic forms.

The aim of this section is modest: to show that, once the local Fisher geometry on coarse-grained densities is fixed by the metriplectic axioms, one can write down an internally consistent scalar channel whose static, weak-field limit reproduces the Newtonian Poisson law, and whose dynamics can be coupled back to the Madelung sector through an effective potential. No new operators are introduced beyond $L_{\rho,G}$, and we stay within a scalar, weak-field, effective description rather than attempting a full tensor theory of gravity.

For clarity, the scalar Fisher sector developed here is an effective, weak-field analogue of Newtonian gravity within the fixed Fisher geometry on densities only. We make no grand assertions.

All scalar-sector identities and examples reported here are reproduced and tested in the accompanying script see:
Appendix F, script `25_fisher_scalar_gravity_checks.py`.

6.1 Static Fisher equation on densities

We work with the scalar case $G = I$ on a spatial slice $\Omega \subset \mathbb{R}^3$ equipped with a strictly positive coarse-grained density ρ and a matter density ρ_{m}. The weighted Poisson operator of Section 6.1.1 becomes

$$L_\rho \varphi := -\nabla \cdot (\rho \nabla \varphi).$$

We postulate that the log density

$$\phi := \log \frac{\rho}{\rho_0},$$

acts as a scalar potential sourced by the matter density via

$$L_\rho \phi = \kappa\, \rho_{\mathrm{m}}, \qquad \kappa > 0,$$

with ρ_0 a reference density and κ a coupling constant. We fix throughout this section a uniform reference density $\rho_0 > 0$; all weak-field scalar expansions below are taken about this constant background.

A short identity reduces (6.1) to an ordinary Poisson equation for ρ.

Lemma 6.1 (*Density form of the Fisher equation*). *For ϕ defined by* (6.1) *one has*

$$\nabla \cdot (\rho \nabla \phi) = \Delta \rho,$$

and hence

$$L_\rho \phi = -\Delta \rho.$$

Proof. Writing $\rho = \rho_0 e^{\phi}$ gives $\nabla\rho = \rho\nabla\phi$ and $\Delta\rho = \nabla\cdot(\rho\nabla\phi)$, which is (6.1). The definition (6.1) then yields $L_\rho\phi = -\nabla\cdot(\rho\nabla\phi) = -\Delta\rho$. □

Combining (6.1) with (6.1) gives

$$\Delta\rho = -\kappa\,\rho_{\mathrm{m}}.$$

Thus the static Fisher equation is equivalent to a Poisson equation for the coarse-grained density, sourced by the matter density.

6.2 Effective potential and Newtonian limit

To compare with Newtonian gravity we define an effective potential

$$\Phi_{eff} := -\frac{c^2}{2}\,\phi = -\frac{c^2}{2}\,\log\frac{\rho}{\rho_0}.$$

The associated acceleration field is

$$g := -\nabla\Phi_{eff} = \frac{c^2}{2}\,\nabla\phi.$$

Using Lemma 6.1 one obtains

$$\nabla\cdot(\rho g) = \frac{c^2}{2}\,\nabla\cdot(\rho\nabla\phi) = \frac{c^2}{2}\,\Delta\rho.$$

In terms of Φ_{eff} the static Fisher equation (6.1) reads

$$-\frac{2}{c^2}\,L_\rho\Phi_{eff} = \kappa\,\rho_{\mathrm{m}}.$$

For later use it is convenient to record the exact relation between Φ_{eff} and ρ. From (6.2),

$$\Delta\Phi_{eff} = -\frac{c^2}{2}\,\Delta\log\frac{\rho}{\rho_0} = -\frac{c^2}{2}\left(\frac{\Delta\rho}{\rho} - \frac{|\nabla\rho|^2}{\rho^2}\right).$$

The second term is quadratic in $\nabla\rho$ and is small when $|\nabla\rho|/\rho$ is small on the scales of interest.

In a weak field, slowly varying regime where ρ is close to a constant background ρ_0 one may write $\rho = \rho_0 + \delta\rho$ with $|\delta\rho| \ll \rho_0$ and $|\nabla\delta\rho|$ small on the scale of interest. To first order in the perturbation,

$$\phi \approx \frac{\delta\rho}{\rho_0}, \qquad \Phi_{eff} \approx -\frac{c^2}{2}\,\frac{\delta\rho}{\rho_0}, \qquad \Delta\Phi_{eff} \approx -\frac{c^2}{2\rho_0}\,\Delta\delta\rho.$$

Using (6.1) with $\rho = \rho_0 + \delta\rho$ and $\Delta\rho_0 = 0$ one has $\Delta\delta\rho = -\kappa\,\rho_{\mathrm{m}}$, so (6.2) yields

$$\Delta\Phi_{eff} \approx \frac{c^2}{2\rho_0}\,\kappa\,\rho_{\rm m}.$$

Matching the Newtonian Poisson equation $\Delta\Phi_N = 4\pi G\rho_{\rm m}$ in this regime fixes

$$\frac{c^2}{2\rho_0}\,\kappa = 4\pi G, \qquad \kappa = \frac{8\pi G}{c^2}\,\rho_0.$$

In this scalar sector we adopt the mass-density interpretation: ρ is a coarse-grained mass density up to an overall constant factor, and the scalar coupling κ absorbs both this factor and the $4\pi G/c^2$ prefactor from the Newtonian Poisson equation. Thus the Fisher scalar sector reproduces the Newtonian Poisson equation for $\Phi_{\rm eff}$ on slowly varying backgrounds after a single calibration of κ against a reference density ρ_0.

Commentary. The weighted operator L_ρ that controls cost, entropy production and curvature also links a log density potential to the Newtonian Poisson equation in a controlled weak field limit.

6.3 Self sourced branch and Fisher star

Even when $\rho_{\rm m}$ is independent, (6.1) admits branches where the coarse grained density itself plays the role of the source. A simple model sets

$$\rho_{\rm m} = \lambda\,\rho, \qquad \lambda > 0,$$

so that (6.1) becomes the Helmholtz equation

$$\Delta\rho + \lambda\kappa\,\rho = 0.$$

For a static spherically symmetric configuration $\rho = \rho(r)$ with $r = |x|$, (6.3) reduces to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\rho}{dr}\right) + \lambda\kappa\,\rho = 0.$$

The regular solution at the origin is

$$\rho(r) = \rho_c\,\frac{\sin(\sqrt{\lambda\kappa}\,r)}{\sqrt{\lambda\kappa}\,r}, \qquad 0 \le r \le R,$$

with central density ρ_c and radius

$$R = \frac{\pi}{\sqrt{\lambda\kappa}}$$

defined by the first zero of the sine. This is the classical $n = 1$ Lane-Emden profile.

The total mass is

$$M = 4\pi\int_0^R \rho_{\rm m}(r)\,r^2 dr = 4\pi\lambda\int_0^R \rho(r)\,r^2 dr = \frac{4\pi\lambda\rho_c}{\sqrt{\lambda\kappa}}\int_0^R r\,\sin(\sqrt{\lambda\kappa}\,r)\,dr.$$

Evaluating the integral explicitly gives

$$M = \frac{4\pi^2 \rho_c}{\kappa^{3/2}\sqrt{\lambda}} = \frac{4\lambda}{\pi}\, \rho_c R^3,$$

Commentary. The self sourced Fisher branch reproduces a standard polytropic profile and ties the radius and mass to the same coupling that appears in the weak field limit. No additional operators are required beyond L_ρ.

6.4 Dynamic Fisher equation and coupling to Madelung

The Madelung companion paper develops a reversible dynamics on a pair (ρ, S) where ρ is a probability density and S a phase. The assembled hydrodynamic fields satisfy a continuity equation and a Hamilton-Jacobi equation regularised by Fisher curvature. It is natural to extend that reversible system by a scalar field ϕ governed by a Fisher-type action built from the same density and weighted operator as in the static scalar sector.

At the level of an effective field theory on a fixed background we may take an action of the schematic form

$$\mathcal{A}[\rho, \phi] = \int dt \int_\Omega \left[\frac{\chi}{2}\, \rho\, (\partial_t \phi)^2 - \frac{\eta c^2}{2}\, \rho\, |\nabla \phi|^2 + \frac{\kappa}{2}\, \rho_{\mathrm{m}}\, \phi \right] dx,$$

with positive constants χ and η. Variation with respect to ϕ yields a dynamic Fisher equation

$$\chi\, \partial_t(\rho\, \partial_t \phi) - \eta c^2\, \nabla \cdot (\rho \nabla \phi) = \frac{\kappa}{2}\, \rho_{\mathrm{m}},$$

or, in terms of the weighted operator L_ρ,

$$\chi\, \partial_t(\rho\, \partial_t \phi) + \eta c^2\, L_\rho \phi = \frac{\kappa}{2}\, \rho_{\mathrm{m}}.$$

Static solutions with $\partial_t \phi = 0$ reduce to

$$L_\rho \phi = \kappa_{eff}\, \rho_{\mathrm{m}}, \qquad \kappa_{eff} := \frac{\kappa}{2\, \eta c^2},$$

so the dynamic sector matches the structure of (6.1) and stays within the same elliptic class used for the cost and curvature identities.

Coupling to the Madelung sector proceeds by taking the same coarse-grained density ρ in the action (6.4) and in the continuity equation, and by allowing the effective potential Φ_{eff} defined in (6.2) to enter the Hamilton-Jacobi equation as an external potential. Schematically, one replaces

$$\partial_t S + \frac{|\nabla S|^2}{2m} + V + Q[\rho] = 0 \quad \text{by} \quad \partial_t S + \frac{|\nabla S|^2}{2m} + V + Q[\rho] + m\, \Phi_{eff} = 0,$$

so that the log-density potential acts on the phase through $m\, \Phi_{eff}$.

Commentary. The reversible Madelung dynamics and the Fisher scalar field can be written over the same density and the same weighted operator L_ρ. The scalar sector is introduced as an additional structure compatible with the existing information geometry, and is kept within a scalar, weak-field, effective description rather than promoted to a separate or complete theory of gravity.

6.5 Microscopic meaning of ρ and Fisher geometry

The density ρ enters both the metriplectic geometry and the Fisher scalar sector as a coarse grained object. A microscopic interpretation can be given in terms of relative entropy between nearby macrostates.

Consider a family of microscopic configurations X with probability measures P_θ indexed by macroscopic parameters $\theta \in \Theta$. Coarse graining to macrostates induces a family of densities ρ_θ on Ω and an associated relative entropy

$$\mathcal{S}(\theta \| \theta') = \int_\Omega \rho_\theta(x) \, \log \frac{\rho_\theta(x)}{\rho_{\theta'}(x)} \, dx.$$

The Fisher information matrix on parameter space is

$$\mathcal{I}_{ab}(\theta) = \int_\Omega \rho_\theta(x) \, \partial_{\theta^a} \log \rho_\theta(x) \, \partial_{\theta^b} \log \rho_\theta(x) \, dx.$$

This is the classical Fisher-Rao information metric on statistical manifolds [37, 38]. In the continuous density picture the quadratic form

$$\mathcal{I}[\varphi] = \int_\Omega \rho(x) \, |\nabla \varphi(x)|^2 dx$$

is the Fisher information of a local perturbation $\delta\rho = \rho\, \varphi$. This is the quadratic form that defines the H_ρ^{-1} metric and the control cost in the metriplectic theory. The same quadratic form appears in the kinetic and gradient terms of the Fisher scalar field.

Commentary. At the microscopic level the Fisher functional that governs dissipation, curvature and control cost is also the object that measures distinguishability between nearby macrostates. In the scalar sector it measures the cost of deforming the log density potential.

6.6 Fisher metric on metrics and the DeWitt quadratic form

The Fisher information construction can be extended to the space of metrics. Consider a family of Gaussian random fields on a background spatial manifold with metric g_{ij}, with covariance controlled by g_{ij}. Perturbing the metric by h_{ij} changes the log likelihood, and the Fisher information on the space of symmetric tensors (h_{ij}) takes the form

$$\mathcal{G}(h,h) = \alpha \int_\Omega \Big(h_{ij} h^{ij} + \beta\, (h^i{}_i)^2 \Big) \sqrt{\det g}\, dx,$$

for suitable constants $\alpha > 0$ and β. This has the same algebraic structure, up to constants, as the DeWitt quadratic form on metric perturbations used in canonical gravity [39].

Commentary. A Fisher metric constructed from a Gaussian field model over a spatial slice has the same algebraic structure as the standard DeWitt quadratic form on metric perturbations. This observation relates a familiar Fisher construction to the configuration space metric used in canonical gravity, without requiring any additional hypotheses.

6.7 Remarks and limitations

The constructions above stay within a scalar sector built from the same density, the operator L_ρ and the Fisher quadratic forms that appear in the metriplectic theory. The static Fisher equation reduces to a Poisson equation for ρ, reproduces the Newtonian Poisson equation for Φ_{eff} in a weak-field, slowly varying regime after a single calibration, and admits self-sourced polytropic profiles with controlled radius and mass. The dynamic Fisher equation and its coupling to the Madelung sector show that the same ingredients can support a scalar field acting on the phase through an effective potential. The emergence of a Newtonian-type potential from information-theoretic structure is in the same broad spirit as entropic gravity proposals [41], although our construction remains strictly within a scalar Fisher sector on coarse-grained densities.

We do not attempt to derive the full Einstein equations, to model realistic equations of state, or to address astrophysical phenomenology beyond simple weak-field and polytropic regimes. Embedding the present scalar channel into a full relativistic framework, or confronting it systematically with data, would require additional structure that lies outside the axioms of this paper. In the same spirit, the Fisher metric on metrics and its DeWitt-type quadratic form are recorded as a geometric echo of standard constructions rather than as a proposal for a new fundamental theory.

The purpose of this section is therefore limited but concrete: once the Fisher geometry on densities is fixed by the metriplectic closure, there is a natural scalar sector aligned with the Newtonian limit that can be written down without introducing further operator machinery. It provides a worked example of how standard gravitational analogies can be expressed within the same information geometry as the dissipative channel, while keeping the scope explicitly effective and scalar.

7 Conclusion

The companion paper isolates the reversible corner and its Fisher curvature; the present work fixes the dissipative channel and records which identities survive locally once G and the boundary class are specified. Within the common information-geometric conventions, the two papers can be read as a single Fisher-regularised information hydrodynamics on (ρ, S): the reversible sector is forced to the Fisher-Schrödinger structure, while the present work determines a compatible metriplectic geometry and its instantaneous diagnostics. Read together, they provide a minimal reversible dissipative split in which one current is decomposed into two quadratures with explicit certificates on both sides, under stated axioms only.

On the dissipative side, a single weighted operator $L_{\rho,G}$ and pairing $\langle\cdot,\cdot\rangle_{\rho,G}$ fix the geometry. The equality dial saturates exactly on the gradient-flow ray $v_{\text{irr}} = \nabla\cdot(\rho G \nabla\mu)$, with $\langle v_{\text{irr}}, \mu\rangle^2 = 2P_{\text{irr}}\dot{\sigma}$ as in the main cost-entropy identity; curvature coercivity controls the local Hessian on the $H_\rho^{-1}(G)$ unit sphere, and the quadratic control cost provides a slice-local reader. These three scalars are instantaneous and insensitive to the reversible bracket at fixed density, and they come with targeted falsifiers that trip as soon as symmetry, ellipticity, positivity, or the tangent model is altered.

On the reversible side, the no-work certificate and $H_\rho^{-1}(G)$ orthogonality characterise the cone defined by antisymmetry and the weighted Liouville identity. In the companion study this structure supports the Fisher curvature and the linear Schrödinger completion, with independent falsifiers built from residual diagnostics, symmetry algebra and superposition tests. In this joint reading, the Schrödinger equation appears as the reversible fixed point of Fisher-regularised information hydrodynamics, while the present paper supplies a metriplectic closure and irreversibility diagnostics that are compatible with it without extending the axioms.

The assembled linear-response and holonomy picture stays local and under control. Linear response obeys a Kramers-Kronig relation under an analyticity guard; sectoriality and ultraviolet behaviour are quantified on a fixed slice; slow loops in control space carry a small geometric phase with integer winding away from branch points. All of these statements are realised by the same KKT machinery and complex reader used for the equality dial. They offer concrete observables that can, in principle, be compared to numerical simulations or experiment-adjacent models in dissipative and quantum settings, without changing the underlying closure.

The Fisher scalar sector fits into the same frame as an optional application slice. Using the weighted operator $L_{\rho,G}$ and the Fisher quadratic form on coarse-grained densities, we recorded a log-density potential with a controlled Newtonian limit, self-sourced polytropic profiles, and a simple dynamic Fisher equation that can be coupled back to the Madelung Hamilton-Jacobi equation as an effective potential. The reversible Madelung dynamics and the Fisher scalar field share the same density and the same weighted operator; the scalar sector is introduced as an additional structure compatible with the existing geometry rather than as a separate theory, and is kept within a scalar, weak-field regime.

Outlook remains local and under guard. A natural next step is to delimit the minimal hypotheses behind the metriplectic closure itself, test additional boundary classes with the same dial-and-modulus readers, and compare with alternative tangent models in discrete and quantum contexts, such as quantum Markov semigroups and Lindblad generators, where entropy-curvature relations take different but related forms. We make no uniqueness or global equivalence assertions. Within scope, the assembly supplies a compact set of slice identities, diagnostics and repairs that render the reversible-dissipative split operationally testable in computation and experiment-adjacent numerics; identifiability of J from F-based scalars alone is not claimed.

Taken together with the reversible analysis of The Converse Madelung Question, the present work provides a minimal reversible-dissipative pair built on a common Fisher geometry. Quantum dynamics, linear response and a simple scalar sector can all be expressed over this shared structure without expanding the axioms, offering a small but coherent step towards an information-geometric view of reversible and irreversible dynamics within the quant-ph and neighbouring communities. Fisher-information and entropic derivations of quantum mechanics [35, 36, 40] provide important context: here the Fisher sector and its Schrödinger completion arise as necessity statements inside a metriplectic and hydrodynamic framework, rather than as independent postulates.

8 Related work and context

Our setting overlaps with classical metriplectic and GENERIC constructions (symmetric G, antisymmetric J, entropy production). Our contributions include:

(i) an equality-saturated cost-entropy inequality in the weighted $H_\rho^{-1}(G)$ geometry,

(ii) curvature coercivity bounds linking Fisher curvature to minimal H^{-1} path cost, and

(iii) operational falsifiers (equality dial, coarse-grain commutator, complex quadratures) that make the metriplectic structure numerically testable on coarse-grained quantum densities.

Metriplectic and GENERIC frameworks ground nonequilibrium thermodynamics [2–5]. Optimal transport and the Otto calculus give the tangent geometry for diffusion, displacement convexity, and curvature bounds, with the dynamic fluid formulation and convexity principles due to Benamou-Brenier and McCann, Otto's porous-medium gradient flow, and AGS as the standard monograph [6–9, 13–16].

Stability of gradient flows under Γ-convergence is classical [17, 18]. Log-Sobolev and transport inequalities support curvature and EVI viewpoints [19–21]. Metric-measure lower curvature provides a complementary framework [22–24]. Our curvature-coercivity estimator in Appendix H tracks observed relaxation rates on the heat-flow oracle and matches the uniform-state Fourier anchor.

Parallel settings include unbalanced transport via Hellinger-Kantorovich when mass is not conserved [27, 28], thermodynamic length for optimal dissipation in control [29–31], and geometric mechanics and double bracket dissipations [32–34]. In our setting the $H_\rho^{-1}(G)$ path cost and length in Sec. 5.5.4 provide the analogous state space geometry for density protocols. The companion paper develops the reversible classification and the role of Fisher curvature with operational falsifiers [1].

A Boundary classes, regularity, and gauges

Domain and admissible boundaries. Let $\Omega \subset \mathbb{R}^d$ be either a d-torus (periodic box) or a bounded Lipschitz domain with outward unit normal n. We work in one of the following boundary classes:

(A1) **Periodic:** fields are periodic and all integration by parts identities hold without boundary terms.

(A2) **No-flux:** the physical flux j satisfies $j \cdot n \equiv 0$ on $\partial\Omega$, hence $\int_\Omega \nabla\cdot j\,dx = 0$.

Unless explicitly stated, all statements in the main text are scoped to (A1) or (A2). Dirichlet or inflow boundaries are outside scope, see Section 4 for the precise identities that fail there. All integration by parts identities used in the paper are justified only within these admissible classes.

State and regularity class. The state is a strictly positive density $\rho : \Omega \to (0, \infty)$ with

$$\rho \in H^1(\Omega), \qquad \int_\Omega \rho\,dx = M > 0, \qquad \rho_{\min} \equiv \operatorname*{ess\,inf}_\Omega \rho \geq \varepsilon > 0.$$

We write $L^2_\#(\Omega)$ for mean zero functions on Ω. The positivity lower bound ε is a scope parameter and appears in the coercivity constants below. In numerics we monitor $\rho_{\min}$ and declare the diagnostics out of scope if $\rho_{\min} < \varepsilon$.

Free energy and chemical potential. The free energy $F[\rho]$ is Fréchet differentiable on the positive cone and defines the chemical potential

$$\mu = \frac{\delta F}{\delta \rho} \quad \text{up to an additive constant.}$$

Only $\nabla\mu$ appears in the dynamics and in the power balances, hence the constant gauge in μ is irrelevant. When needed, we fix $\int_\Omega \mu\,dx = 0$.

Weighted Poisson operator and coercivity. Define the weighted Poisson operator $\mathcal{L}_\rho : H^1(\Omega)/\mathbb{R} \to H^{-1}(\Omega)$ by

$$\mathcal{L}_\rho \phi \;=\; -\nabla\cdot(\rho\,\nabla\phi),$$

with domain consisting of mean zero H^1 functions in the periodic case, and of H^1 functions with $\nabla\phi \cdot n = 0$ in the no-flux case. For ϕ, ψ in the domain, integration by parts yields

$$\langle \phi, \mathcal{L}_\rho \psi \rangle \;=\; \int_\Omega \rho\,\nabla\phi \cdot \nabla\psi\,dx \;=\; \langle \psi, \mathcal{L}_\rho \phi \rangle,$$

so $\mathcal{L}_\rho$ is symmetric and nonnegative. Moreover,

$$\int_\Omega \rho\,|\nabla\phi|^2\,dx \;\geq\; \rho_{\min} \int_\Omega |\nabla\phi|^2\,dx \;\geq\; c_P(\Omega)\,\rho_{\min}\,\|\phi\|^2_{H^1/\mathbb{R}},$$

where $c_P(\Omega) > 0$ is a Poincaré constant that depends only on Ω and the boundary class. Hence $\mathcal{L}_\rho$ is coercive on mean zero potentials, with coercivity constant proportional to $\rho_{\min}$.

$\mathbf{H^{-1}}$ pairing and uniqueness of potentials. For $v \in H^{-1}(\Omega)$ with zero mean, the Riesz map induced by $\mathcal{L}_\rho$ defines the weighted H^{-1} norm

$$\|v\|^2_{H^{-1}(\rho)} \;=\; \inf_{\phi \in H^1/\mathbb{R}} \left\{ \int_\Omega \rho\, |\nabla \phi|^2 \, dx \;\;:\;\; v = \nabla\cdot(\rho\, \nabla \phi) \right\}.$$

Coercivity in (A) implies existence and uniqueness of the potential ϕ solving $\nabla\cdot(\rho\,\nabla\phi) = v$, modulo constants. We always fix the mean-zero gauge on ϕ. This is the unique potential used in the cost functional and in duality estimates.

Conservative divergence form and mass conservation. Let $j = \rho\, u$ be any flux with $u \in L^2(\Omega)^d$. The conservative update is $\partial_t \rho = -\nabla\cdot j$. In classes (A1) and (A2),

$$\frac{d}{dt}\int_\Omega \rho\, dx \;=\; -\int_\Omega \nabla\cdot j\, dx \;=\; -\int_{\partial\Omega} j\cdot n\, dS \;=\; 0,$$

so total mass is conserved. All variational statements in the main text are written in divergence form to preserve this identity at the discrete level as well.

Irreversible and reversible fluxes. Within the admissible class, the irreversible flux has the form

$$j_{\mathrm{irr}} \;=\; -\,\rho\, G(\rho, x)\, \nabla \mu,$$

where G is a bounded, symmetric, positive operator that acts locally at each point x and depends on ρ only through its value at x. The reversible flux j_{rev} is

$$j_{\mathrm{rev}} \;=\; -\,\rho\, J(\rho, x)\, \nabla \mu,$$

and it satisfies the scalar no-work identity

$$\int_\Omega \mu\, \nabla\cdot(\rho\, J\, \nabla \mu)\, dx = 0 \quad \text{for all smooth } \mu,$$

which holds if $J^\top = -J$ and $\nabla\cdot(\rho J) = 0$ (Appendix B). In both cases, boundary class (A1) or (A2) ensures compatibility with the conservative form.

Work, dissipation, and the equality case. The instantaneous irreversible power and entropy production at a fixed state ρ are

$$P_{\mathrm{irr}}(\rho;\mu) \;=\; \frac{1}{2}\int_\Omega \rho\, (\nabla\mu)\cdot G(\rho,x)\, (\nabla \mu)\, dx, \qquad \dot\sigma(\rho) \;=\; -\frac{d}{dt}F[\rho] \;=\; -\int_\Omega \mu\, \partial_t \rho\, dx.$$

For the realised irreversible direction $v_{\text{irr}} = \nabla\cdot(\rho\, G\nabla\mu)$ one has, on periodic or no-flux boundaries,

$$\langle v_{\text{irr}},\, \mu\rangle \;=\; -\,2\,P_{\text{irr}}(\rho;\mu), \qquad \dot\sigma(\rho) \;=\; 2\,P_{\text{irr}}(\rho;\mu).$$

Consequently the sharp Cauchy-Schwarz equality reads

$$\langle v_{\text{irr}},\, \mu\rangle^2 \;=\; 2\,P_{\text{irr}}(\rho;\mu)\,\dot\sigma(\rho).$$

Here $\langle\cdot,\cdot\rangle$ is the L^2 pairing on Ω. For any other flux with the same power, the left side is strictly smaller. Identity (A) and the no-work condition for the reversible flux, $\langle v_J,\, \mu\rangle = 0$, are the two scalar certificates used in the diagnostics.

Discrete realisation and tripwires. In the spectral code we implement the 2/3 projector P on all nonlinear operations to avoid aliasing and we compute the mass integral from the zero Fourier mode. We evaluate P_{irr} using the projected gradient to match the subspace of v_{irr} and we enforce the boundary class by construction. The diagnostics report:

(D1) $\int_\Omega \partial_t\rho\, dx = 0$ to machine precision,
(D2) the equality gap $2P_{\text{irr}}\sigma - \langle v_{\text{irr}}, \mu\rangle^2$ decreases with mesh refinement,
(D3) the reversible power $\int_\Omega \rho\,\nabla\mu\cdot J\nabla\mu\, dx$ is at numerical zero.

Any violation of these tripwires indicates a departure from the boundary classes or regularity stated above.

Scope guard for vacua. If $\rho_{\min} \downarrow 0$ the coercivity constant in (A) degenerates and the potential ϕ becomes nonunique across components where ρ vanishes. Our scope is restricted to $\rho_{\min} \geq \varepsilon$. Weak solutions with vacua can be treated by working on connected components of $\{\rho > 0\}$ and by fixing the gauge of ϕ on each component. This regime is outside the assertions of the main theorems but can be diagnosed by monitoring $\rho_{\min}$.

B Reversible no-work, weighted Liouville form, and orthogonality

Aim. We record precise conditions under which the reversible channel performs no-work on the free energy and exhibits an exact orthogonality with the irreversible channel. The key object is a weighted Liouville structure that depends on the state ρ.

Setting. Work on the domain and boundary classes of Appendix A. Let $F[\rho]$ be Fréchet differentiable on the positive cone, with chemical potential $\mu = \delta F/\delta\rho$ determined up to an additive constant. A reversible flux has the general form

$$j_{\text{rev}}(\rho,\mu) \;=\; -\,\rho\, J(\rho,x)\,\nabla\mu,$$

The reversible generator is the conservative update

$$\partial_t \rho\big|_{\text{rev}} \;=\; -\nabla\cdot j_{\text{rev}} \;=\; \nabla\cdot\big(\rho\, J(\rho,x)\,\nabla\mu\big).$$

No-work condition. Define the reversible power

$$P_{\text{rev}}(\rho;\mu) \;=\; \int_\Omega \mu\,\partial_t\rho\big|_{\text{rev}}\,dx \;=\; -\int_\Omega \mu\,\nabla\cdot(\rho\,J\,\nabla\mu)\,dx.$$

Integration by parts within the boundary classes and the antisymmetry of J yield the algebraic identity below.

Lemma B.1 (*no-work equivalence*). *Assume $J(\rho,x)$ is pointwise antisymmetric, $J^\top = -J$, and satisfies the weighted Liouville identity*

$$\nabla\cdot\big(\rho\,J(\rho,x)\big) = 0 \quad \textit{in the sense of distributions.}$$

Equivalently, each column of the matrix field ρJ is divergence-free, i.e. ρJ is solenoidal with respect to the Lebesgue measure.

Then $P_{\text{rev}}(\rho;\mu) = 0$ for all smooth μ with any constant gauge, hence along reversible trajectories $F[\rho(t)]$ is constant.

Conversely, if $P_{\text{rev}}(\rho;\mu) = 0$ for all smooth μ and all states ρ in the admissible class, then J must be antisymmetric and satisfy (B.1).

Proof. Under the assumptions (and working with the unique mean-zero KKT potential on the positive cone),

$$P_{\text{rev}} = -\int_\Omega \mu\,\nabla\cdot(\rho J\nabla\mu)\,dx = \int_\Omega \rho\,(\nabla\mu)^\top J\,\nabla\mu\,dx \;+\; \int_\Omega \mu\,(\nabla\cdot(\rho J))\cdot\nabla\mu\,dx.$$

The second integral vanishes by (B.1). The first vanishes pointwise since $a^\top J a = 0$ for all vectors a if $J^\top = -J$. For the converse, take test potentials of the form $\mu = \phi + \varepsilon\psi$ and evaluate P_{rev} at several choices; linear independence in ϕ, ψ forces antisymmetry of J and (B.1). Details are standard and omitted. □

Weighted bracket Define the reversible generator on functionals $A[\rho]$ via

$$\{A,F\}_\rho \;\equiv\; \int_\Omega \frac{\delta A}{\delta\rho}\,\nabla\cdot(\rho\,J\,\nabla\mu)\,dx \;=\; -\int_\Omega \rho\,\nabla\Big(\frac{\delta A}{\delta\rho}\Big)\cdot J\,\nabla\mu\,dx,$$

with $\mu = \delta F/\delta\rho$ and where the second equality uses (B.1) to remove a boundary term. Under Lemma B.1, the bracket is skew:

$$\{A,F\}_\rho \;=\; -\{F,A\}_\rho.$$

We do not require the full Jacobi identity for the results in the main text. The only properties used are skewness, Leibniz, and that $\{F,F\}_\rho \equiv 0$.

Orthogonality of reversible and irreversible channels. Let the irreversible direction be $v_{\mathrm{irr}} = \nabla\cdot(\rho\, G\, \nabla\mu)$ with $G = G^\top \succ 0$ local. Consider the weighted H^{-1} pairing induced by $\mathcal{L}_\rho = -\nabla\cdot(\rho\nabla\cdot)$, as in Appendix A. Denote by ϕ the unique mean-zero potential solving $-\mathcal{L}_\rho\phi = v_{\mathrm{irr}}$. Then

$$\begin{aligned}\langle v_{\mathrm{rev}}, \phi\rangle &= \int_\Omega \phi\, \nabla\cdot(\rho\, J\, \nabla\mu)\, dx = -\int_\Omega \rho\, \nabla\phi \cdot J\, \nabla\mu\, dx \\ &= \int_\Omega \rho\, \nabla\mu \cdot J\, \nabla\phi\, dx = \int_\Omega \mu\, \nabla\cdot(\rho\, J\, \nabla\phi)\, dx = 0,\end{aligned}$$

where the third equality uses antisymmetry and the last uses (B.1). Hence v_{rev} lies in the H^{-1}_ρ orthogonal complement of the irreversible cone. This establishes the metriplectic orthogonality used in the equality case and in the diagnostics.

Uniqueness of the reversible class within the no-work cone. Suppose J_1 and J_2 are antisymmetric and satisfy (B.1). Then for any μ,

$$\int_\Omega \mu\, \nabla\cdot\big(\rho\, (J_1 - J_2)\, \nabla\mu\big)\, dx = 0.$$

Hence J_1 and J_2 generate the same scalar invariants on F. Differences between reversible generators that maintain (B.1) are invisible to the equality dial. Identifiability of J requires additional observables beyond F; we do not assert such identifiability in the main text.

Failure modes and tripwires. If either hypothesis of Lemma B.1 fails, the scalar certificate breaks in a controlled way:

- If $J^\top \neq -J$, then pointwise $a^\top J a$ can be nonzero and P_{rev} picks up a bulk term.
- If $\nabla\cdot(\rho J) \neq 0$, then even for antisymmetric J a boundary-free bulk term survives:
$$P_{\mathrm{rev}} = \int_\Omega \mu\, \nabla\cdot(\rho J)\, \nabla\mu\, dx,$$
which is generically nonzero. In the spectral code this appears as a reversible power at $\mathcal{O}(1)$ relative scale, so the PR dial triggers.
- If boundaries violate the admissible classes, integration by parts produces boundary work of the form $\int_{\partial\Omega} \mu\, \rho(J\nabla\mu)\cdot n\, dS$, which is detected by the mass and equality dials.

Discrete realisation. In the diagnostics we implement two checks:

(R1) **Power dial:** evaluate $P_{\mathrm{rev}} = \int \rho\, \nabla\mu \cdot J\nabla\mu\, dx$ at machine zero by using a Liouville-compatible field $J_\rho = c\,\rho^{-1}\varepsilon$ (with constant c and fixed antisymmetric ε), so that $\nabla\cdot(\rho J_\rho) = 0$ holds exactly on the grid.

(R2) **Divergence dial:** evaluate $\|\nabla\cdot(\rho J_\rho \nabla\mu)\|_2$, which vanishes to roundoff for J_ρ as above, certifying the algebra; departures scale with $\|\nabla J\|$ when J varies in space and decay under refinement.

When J varies in x, both dials remain valid but no longer vanish exactly; they scale with the size of ∇J and with grid refinement in a way consistent with Appendix C.

Relation to Hamiltonian hydrodynamics. Condition (B.1) is the density-weighted analogue of a divergence-free Hamiltonian flow in canonical variables. The bracket (B) is the natural pushforward of the canonical bracket under the Madelung map when restricted to functionals of ρ alone. Our results rely only on skewness and no-work, not on a full Jacobi structure on the space of densities.

Summary. The reversible class that performs no-work on F is characterised by the weighted Liouville identity (B.1) together with antisymmetry of J. Under these hypotheses the reversible and irreversible channels are orthogonal in the weighted H^{-1} pairing, the scalar power certificate $P_{\mathrm{rev}} = 0$ holds for all μ, and the diagnostics in Section G report machine-zero values in the constant J case, with controlled departures when J varies smoothly in space.

C Coarse-graining and commutator

Reported commutator dial. We compute $\mathrm{rel}(\ell) = \|C_\ell(Q(\rho)) - Q(C_\ell\rho)\|_{L^2}$ and $\mathrm{rel}(\ell)/\ell^2$. All nonlinear products are 2/3-dealiased, and the same Gaussian filter is applied before norms so pairings live in the same spectral subspace.

Aim. We quantify how a microscopic irreversible generator fails to commute with coarse-graining at small filter width ℓ. The main statement is that the commutator between the coarse-graining operator C_ℓ and the irreversible evolution is of order ℓ^2 on smooth states, with an explicit leading-order structure.

Set-up and notation. Let Ω be either a periodic box or a bounded Lipschitz domain with no-flux boundaries, as in Appendix A. Let the irreversible generator be

$$\partial_t \rho \;=\; \mathcal{Q}(\rho) \;=\; \nabla\cdot\big(\rho\, G(\rho, x)\, \nabla\mu(\rho)\big), \qquad \mu(\rho) \equiv \frac{\delta F}{\delta \rho},$$

with G bounded, symmetric, positive and local in x, and F Fréchet differentiable on the positive cone. For concreteness, many examples in the text take $F[\rho] = \int \rho(\log\rho - 1)\, dx + \int U(\rho, x)\, dx$ with U smooth in ρ and x.

We define the coarse-graining operator C_ℓ as convolution with a centred, isotropic mollifier K_ℓ of width $\ell > 0$, normalised to unit mass, and with vanishing first moments. For a Gaussian filter,

$$(C_\ell f)(x) = \int_\Omega K_\ell(x-y) f(y)\, dy, \qquad \widehat{C_\ell f}(k) = e^{-\frac{1}{2}\ell^2|k|^2}\, \hat{f}(k).$$

Moment relations give, for smooth f,

Regularity and constants. Throughout this appendix we assume $\rho \in H^3(\Omega)$ with $\rho_{\min} > 0$, $A(\rho, x) = \rho\, G(\rho, x)$ and $\mu(\rho)$ are C^2 in ρ and smooth in x, and G is uniformly elliptic with bounds $0 < \gamma_{\min} \le \xi^\top G(\rho, x)\, \xi \le \gamma_{\max} < \infty$. All K-constants below depend only on $(\rho_{\min}, \gamma_{\min}, \gamma_{\max})$ and the H^3 norm of ρ on the stated domain and boundary class.

$$C_\ell f \;=\; f + \frac{\ell^2}{2}\,\Delta f + \mathcal{O}(\ell^4) \quad \text{in } L^2.$$

Commutator. We study

$$\mathcal{R}_\ell(\rho) \;\equiv\; C_\ell(\mathcal{Q}(\rho)) \;-\; \mathcal{Q}(C_\ell \rho).$$

The first term evolves then coarse-grains. The second coarse-grains then evolves. The assertion is that $\|\mathcal{R}_\ell(\rho)\|_{L^2}$ scales like ℓ^2 for smooth ρ with $\rho_{\min} > 0$.

Lemma C.1 (*Local expansion*). *Let $A(\rho, x) \equiv \rho\, G(\rho, x)$ and write $\mathcal{Q}(\rho) = \nabla\cdot\big(A(\rho, x)\, \nabla \mu(\rho)\big)$. Assume A and μ are C^2 in ρ and smooth in x on the positive cone. Then*

$$\begin{aligned}\mathcal{R}_\ell(\rho) &= C_\ell(\nabla\cdot(A\nabla\mu)) \;-\; \nabla\cdot\Big(A(\rho_\ell, x)\, \nabla\mu(\rho_\ell)\Big), \quad \rho_\ell := C_\ell \rho \\ &= \frac{\ell^2}{2}\Big[\Delta\, \nabla\cdot(A\nabla\mu) \;-\; \nabla\cdot\big((\partial_\rho A)\, \Delta\rho\, \nabla\mu \;+\; A\, \nabla(\partial_\rho\mu\, \Delta\rho)\big)\Big] \;+\; \mathcal{O}(\ell^4),\end{aligned}$$

with all quantities evaluated at (ρ, x) and where $\partial_\rho A$ and $\partial_\rho \mu$ denote Fréchet derivatives applied to $\Delta\rho$ in the direction of the Laplacian perturbation coming from (C).

Sketch. Apply (C) to C_ℓ acting on scalars and vector fields, and to the compositions $A(\rho_\ell, x)$ and $\mu(\rho_\ell)$ via first-order Taylor in $\rho_\ell - \rho = \frac{\ell^2}{2}\Delta\rho + \mathcal{O}(\ell^4)$. Then expand $\nabla\cdot(A\nabla\mu)$ at (ρ_ℓ, x) to the same order. Collecting terms yields (C.1). Regularity and $\rho_{\min} > 0$ ensure all coefficients are bounded. □

Proposition C.2 (*Order ℓ^2 commutator*). *Under the assumptions above there exists $K = K(\rho, G, F, \Omega) > 0$ such that, for ℓ small,*

$$\|\mathcal{R}_\ell(\rho)\|_{L^2} \;\le\; K\, \ell^2 \Big(\|\rho\|_{H^3} + 1\Big), \qquad \textit{hence} \qquad \frac{\|\mathcal{R}_\ell(\rho)\|_{L^2}}{\|\mathcal{Q}(\rho)\|_{L^2}} \;=\; \mathcal{O}(\ell^2).$$

Proof. By Lemma C.1 the leading remainder is a linear combination of terms with three spatial derivatives falling on ρ and $\mu(\rho)$, with coefficients bounded on the positive cone by smoothness of A and μ. Standard product estimates in H^s on Lipschitz domains then give (C.2). The denominator is nonzero for nontrivial states away from equilibrium. □

Interpretation. Coarse-graining and evolving do not commute, but the defect is $\mathcal{O}(\ell^2)$ for smooth states under the stated bounds. The ℓ^2 law is certified numerically, both in absolute and relative form.

Reversible contribution. If the reversible flux is $j_{\mathrm{rev}} = -\,\rho\, J(x)\,\nabla\mu$ with $\nabla\cdot(\rho J) = 0$, then

$$\partial_t \rho\big|_{\mathrm{rev}} = \nabla\cdot(\rho J \nabla\mu).$$

If J is constant in space, C_ℓ and the reversible generator commute exactly on periodic domains (and on no-flux boxes for filters supported away from the boundary), since convolution commutes with constant-coefficient differential operators in the interior. If J varies smoothly in x, an expansion identical in spirit to Lemma C.1 shows a defect of order ℓ^2 with coefficients involving ∇J and $\nabla^2 J$, bounded by the same regularity and ellipticity constants.

Discrete normalisation and the Reported commutator dial. In the diagnostics we report the relative commutator

$$\mathrm{rel}(\ell) \;=\; \frac{\|C_\ell(\mathcal{Q}(\rho)) - \mathcal{Q}(C_\ell\rho)\|_{L^2}}{\|\mathcal{Q}(\rho)\|_{L^2}}, \qquad \text{and the ratio} \qquad \frac{\mathrm{rel}(\ell)}{\ell^2}.$$

The coarse-graining C_ℓ is implemented spectrally as multiplication by $e^{-\frac{1}{2}\ell^2|k|^2}$. We use the same Gaussian C^∞ mollifier for all runs; top-hat filters were tested and found to spoil the observed ℓ^2 law, as expected. To avoid aliasing we apply the 2/3 projector P to all nonlinear products before computing $\mathcal{Q}(\rho)$ and again to the outputs, so that all pairings and norms live in the same spectral subspace. The empirical observation in Section G is that $\mathrm{rel}(\ell)/\ell^2$ remains in a narrow band for small ℓ, consistent with Proposition C.2.

Leading-order drift under coarse-graining. Although (C.2) suffices for the dial, it is useful to record the induced drift on the operator. Writing $C_\ell\mathcal{Q}(\rho) = \mathcal{Q}(\rho) + \frac{\ell^2}{2}\Delta\mathcal{Q}(\rho) + \mathcal{O}(\ell^4)$ and expanding $\mathcal{Q}(C_\ell\rho)$ via Lemma C.1, one finds

$$\mathcal{Q}(C_\ell\rho) \;=\; \nabla\!\cdot\!\Big(\rho\, G\,\nabla\mu\Big) + \frac{\ell^2}{2}\,\nabla\!\cdot\!\Big(\underbrace{(\partial_\rho(\rho G))\,\Delta\rho\,\nabla\mu \;+\; \rho G\,\nabla(\partial_\rho\mu\,\Delta\rho)}_{\text{closure drift}}\Big) + \mathcal{O}(\ell^4).$$

Thus, to leading order, coarse-graining renormalises the irreversible operator by a correction quadratic in $\nabla\rho$ and linear in $\Delta\rho$, with coefficients controlled by $(\rho_{\min}, \gamma_{\min}, \gamma_{\max})$; this matches the observed stability of the equality dial under mild filtering.

Reversible generator. For smooth antisymmetric J and Gaussian coarse-graining,

$$J_\ell \;=\; J \;+\; \tfrac{1}{2}\,\ell^2\,\Delta J \;+\; O(\ell^4),$$

with the $O(\ell^4)$ residual verified numerically on the same grids and Gaussian filters used for the commutator dial.

Scope and limitations. The analysis above relies on smoothness and on locality of G and μ. If F contains nonlocal interactions via convolution kernels or if G encodes finite-range hydrodynamic couplings, the same machinery applies with extra commutator terms involving the kernel length scale. In that case, the normalised dial remains meaningful but the ℓ^2 law can cross over to a mixed law in ℓ and the nonlocal range.

Numerical check. For the entropy-only case with $G(x) = 1 + 0.4\cos(2x)$ we observe

$$\frac{\mathrm{rel}(\ell)}{\ell^2} \approx \text{constant for small } \ell,$$

with weak dependence on resolution after de-aliasing and projection. This matches Proposition C.2 and validates the use of the commutator dial as a guard for form stability under mild coarse-graining.

D Necessity chain and short proofs

We now summarise the logical flow from the seven axioms (A1-A7) to the local metriplectic structure, giving short proofs of each link and identifying the scalar certificates that lock the geometry.

Equality dial refers to the irreversible scalar certificate; PR dial reports the reversible power; the commutator dial is defined in Appendix C.

D.1 From A1-A4 to the weighted H^{-1} tangent

A1 (mass conservation) and A4 (probe locality) ensure that all admissible variations of ρ occur through conservative directions $v = -\nabla\cdot(\rho u)$ with $u \in L^2(\Omega)^d$. Defining the potential ϕ by $u = -\nabla\phi$ gives the weighted Poisson operator

$$\mathcal{L}_\rho\phi = -\nabla\cdot(\rho\nabla\phi),$$

symmetric and coercive on mean-zero H^1 functions (Appendix A). Every admissible v can thus be represented uniquely as $v = -\mathcal{L}_\rho\phi$ up to constants.

This establishes the weighted H^{-1}_ρ tangent space and provides the setting for the quadratic form of A3.

D.2 From A3-A5 to the irreversible generator

A3 postulates a local quadratic power

$$P_{\mathrm{irr}}(\rho;\mu) = \tfrac{1}{2}\int_\Omega \rho\,(\nabla\mu)\cdot G(\rho,x)\,(\nabla\mu)\,dx.$$

A5 (steepest descent) requires that the realised $v_{\rm irr}$ maximises $\sigma = -\langle v, \mu\rangle$ subject to fixed $P_{\rm irr}$. By Cauchy-Schwarz in the ρ-weighted G^{-1} metric one obtains

$$\langle v, \mu\rangle^2 \le 2\, P_{\rm irr}\, \dot\sigma(\rho),$$

with equality only for $v = \nabla\cdot(\rho\, G\nabla\mu)$. Hence

$$v_{\rm irr} = \nabla\cdot(\rho\, G\, \nabla\mu),$$

and the equality certificate

$$\langle v_{\rm irr}, \mu\rangle^2 = 2\, P_{\rm irr}\, \dot\sigma(\rho)$$

which is verified numerically in Appendix G, lines `N=256-4096 | PASS`. Any modification of G to include nonlocal coupling or non-quadratic terms breaks this identity, as shown by the falsifier sweep. Thus the local quadratic form and steepest-descent rule are not assumptions but necessities within A1-A5.

D.3 From A6 to the reversible class

A6 demands that the reversible channel perform no-work on F, i.e. $P_{\rm rev} = 0$ for all μ. Lemma B.1 (Appendix B) shows this is equivalent to

$$j_{\rm rev} = -\,\rho\, J(\rho, x)\, \nabla\mu, \qquad J^\top = -J, \qquad \nabla\cdot(\rho J) = 0.$$

The second condition enforces antisymmetry; the third is the weighted Liouville identity. Conversely, these imply $P_{\rm rev} = 0$ identically, producing the reversible no-work identity

$$\int_\Omega \rho\, (\nabla\mu)\cdot J\, (\nabla\mu)\, dx = 0,$$

verified in the diagnostics as `PR = 0.000e+00 | PASS`. Hence A6 singles out a unique orthogonal complement to the irreversible cone.

D.4 Orthogonality and metriplectic closure

Let ϕ solve $-\,\mathcal{L}_\rho\phi = v_{\rm irr}$. Using antisymmetry of J and $\nabla\cdot(\rho J) = 0$,

$$\langle v_{\rm rev}, \phi\rangle = \int_\Omega \phi\, \nabla\cdot(\rho J \nabla\mu)\, dx = -\int_\Omega \rho\, \nabla\phi\cdot J\, \nabla\mu\, dx = 0.$$

Thus the reversible and irreversible directions are H^{-1}_ρ-orthogonal. Together, they define the metriplectic decomposition

$$\partial_t\rho = \nabla\cdot(\rho\, G\, \nabla\mu) + \nabla\cdot(\rho\, J\, \nabla\mu),$$

with $G = G^\top \succ 0$, $J^\top = -J$, and $\nabla\cdot(\rho J) = 0$. The irreversible channel satisfies the equality certificate, the reversible channel the no-work certificate, and both preserve total mass.

D.5 Verification chain in diagnostics

> *Remark* (Four checkpoints). Equality on the irreversible ray, no-work on the equality dial, orthogonality in H_ρ^{-1}, and conservative mass with exact DC pinning form a minimal pass-fail chain. Any break trips immediately.

We use equality dial uniformly for the instantaneous scalar diagnostic of the no-work or equality condition.

- **Equality dial:** confirms Proposition 2.2 and the steepest-descent equality.
- **Nonlocal falsifier:** breaks the equality, confirming necessity of A3.
- **Reversible PR dial:** reports machine-zero power, confirming A6.
- **Orthogonality:** follows analytically and is indirectly checked by the simultaneous success of the previous two dials.

Together these complete the necessity chain: every axiom has a direct empirical or algebraic certificate, and every certificate fails immediately when an axiom is relaxed.

E Operator facts

Wasserstein tangent, Poisson operator, and coercivity

Define $\mathcal{L}_\rho \phi = -\nabla\cdot(\rho\nabla\phi)$ on the domain of mean zero H^1 functions with periodic or no-flux boundaries. For mean zero $\phi, \psi \in H^1$,

$$\langle \phi, \mathcal{L}_\rho \psi\rangle = \int_\Omega \rho\,\nabla\phi\cdot\nabla\psi\,dx = \langle \psi, \mathcal{L}_\rho \phi\rangle,$$

so $\mathcal{L}_\rho$ is symmetric and positive on the mean-zero subspace. Coercivity follows from $\int_\Omega \rho\,|\nabla\phi|^2\,dx \geq \rho_{\min}\int_\Omega |\nabla\phi|^2\,dx$. The H_ρ^{-1} inner product is

$$\langle a, b\rangle_{H_\rho^{-1}} \equiv \int_\Omega \rho\,\nabla\phi_a\cdot\nabla\phi_b\,dx, \quad -\nabla\cdot(\rho\nabla\phi_a) = a, \quad -\nabla\cdot(\rho\nabla\phi_b) = b,$$

which is well defined on mean zero tangents [6–9].

KKT characterisation of minimal cost

Given $v = \nabla\cdot(\rho u)$ define the functional

$$\mathcal{J}[u,\phi] = \frac{1}{2}\int_\Omega \rho\,u^\top G^{-1} u\,dx + \int_\Omega \phi(\nabla\cdot(\rho u) - v)\,dx.$$

Stationarity in u gives $G^{-1}u - \nabla\phi = 0$, hence $u^\star = G\nabla\phi$. The constraint gives $-L_{\rho,G}\phi = v$, where $L_{\rho,G}\phi \equiv -\nabla\cdot(\rho\, G\nabla\phi)$. Substituting back yields

$$\mathcal{C}_{\min}(\rho; v) = \frac{1}{2}\int_\Omega \rho\,(\nabla\phi)^\top G(\nabla\phi)\,dx,$$

with constants controlled by ellipticity bounds $\gamma_{\min}, \gamma_{\max}$ and the positivity margin $\rho_{\min}$.

Proof of Proposition 3.1 (cost-entropy inequality)

Let $v = \nabla\cdot(\rho u)$ and define the ρ-weighted pairing $\langle a, b\rangle_\rho = \int_\Omega \rho\, a\cdot b\,dx$. Integration by parts yields

$$\langle v, \mu\rangle = -\int_\Omega \rho\, u\cdot\nabla\mu\,dx = \langle u,\, G^{-1}(G\nabla\mu)\rangle_\rho.$$

Cauchy Schwarz in the G^{-1} metric gives

$$\langle v, \mu\rangle^2 \le \Big(\int_\Omega \rho\, u^\top G^{-1} u\,dx\Big)\Big(\int_\Omega \rho\,(\nabla\mu)^\top G(\nabla\mu)\,dx\Big) = 2\,\mathcal{C}(u)\,\dot\sigma(\rho).$$

Minimising over admissible u gives the stated lower bound for $\mathcal{C}_{\min}(\rho; v)$. Equality holds if and only if u is everywhere collinear with $G\nabla\mu$.

Proof of curvature coercivity

Let $v = -\nabla\cdot(\rho\nabla\psi)$ with mean zero ψ and normalise $\|v\|^2_{H^{-1}_\rho} = \int_\Omega \rho\,|\nabla\psi|^2\,dx = 1$. Relate this to the quadratic form that defines $\mathcal{C}_{\min}$ by noting that for φ solving $-\nabla\cdot(\rho\nabla\varphi) = v$ (the H^{-1}_ρ potential),

$$\int_\Omega \rho\,(\nabla\varphi)^\top G(\nabla\varphi)\,dx \le \gamma_{\max}\int_\Omega \rho\,|\nabla\varphi|^2\,dx = \gamma_{\max}\,\|v\|^2_{H^{-1}_\rho},$$

and the reverse inequality uses $\gamma_{\min}$.

We recall the Rayleigh formulation

$$\kappa_{\min}(\rho) \;=\; \inf_{v\neq 0} \frac{\langle \mathcal{H}_F(\rho)\,v,\, v\rangle}{\|v\|^2_{H^{-1}_\rho}}\,.$$

By definition of $\kappa_{\min}$ one has the sharp bound

$$\langle \mathcal{H}_F(\rho)\,v,\, v\rangle \;\ge\; \kappa_{\min}(\rho)\,\|v\|^2_{H^{-1}_\rho}.$$

Using the ellipticity bounds

$$\gamma_{\min}\int_\Omega \rho\,|\nabla\varphi|^2\,dx \;\le\; \int_\Omega \rho\,(\nabla\varphi)^\top G(\nabla\varphi)\,dx \;\le\; \gamma_{\max}\int_\Omega \rho\,|\nabla\varphi|^2\,dx,$$

and $2\,\mathcal{C}_{\min}(\rho;v) = \int_\Omega \rho\,(\nabla\varphi)^\top G(\nabla\varphi)\,dx$, we obtain the corollary

$$\langle \mathcal{H}_F(\rho)\,v,\,v\rangle \;\ge\; \frac{\kappa_{\min}(\rho)}{\gamma_{\max}}\cdot 2\,\mathcal{C}_{\min}(\rho;v).$$

Proof of Lemma 3.4 (alignment identity in the (ρ, G) metric)

Let ϕ solve $-L_{\rho,G}\phi = v$ with $L_{\rho,G}\phi \equiv -\nabla\cdot(\rho G\nabla\phi)$. Define the (ρ, G) inner product on vector fields by

$$\langle a, b\rangle_{\rho,G} \equiv \int_\Omega \rho\,a^\top G\,b\,dx, \qquad \|a\|^2_{\rho,G} = \langle a, a\rangle_{\rho,G}.$$

For the minimiser $u^\star = G\nabla\phi$ (by A.2) one has

$$\langle v, \mu\rangle = -\int_\Omega \rho\,u^\star\cdot\nabla\mu\,dx = -\langle\nabla\phi, \nabla\mu\rangle_{\rho,G}.$$

Moreover,

$$2\,\mathcal{C}_{\min}(\rho;v) = \|\nabla\phi\|^2_{\rho,G}, \qquad \dot\sigma(\rho) = \|\nabla\mu\|^2_{\rho,G}.$$

Hence

$$\mathcal{R}(\rho;v) \equiv \frac{\langle v,\mu\rangle^2}{2\,\mathcal{C}_{\min}(\rho;v)\,\dot\sigma(\rho)} = \frac{\langle\nabla\phi,\nabla\mu\rangle^2_{\rho,G}}{\|\nabla\phi\|^2_{\rho,G}\,\|\nabla\mu\|^2_{\rho,G}} = \cos^2\theta_{\rho,G} \in [0,1],$$

with $\mathcal{R} = 1$ if and only if $\nabla\phi$ is collinear with $\nabla\mu$ (equivalently $u^\star \parallel G\nabla\mu$). This proves Lemma 3.4.

F Code archive

All numerical checks are performed using short, self-contained Python scripts hosted at:

`https://github.com/feuras/metriplectic/`

The scripts implement direct numerical tests of the metriplectic structure, dissipation identities, and equivalence statements discussed in the main text. Each test is designed to be fully reproducible using only `NumPy`, `SciPy`, and `pandas`, and all produce text-only console outputs. The code does not rely on any external packages or plotting tools.

Structure and scope

- **0A__axiom__diagnostics.py**
A consolidated, review-ready dial suite: EVI probe with a saturation readout on the irreversible ray; Noether no-work symmetry showing invariance of $\dot{\sigma}$, $\kappa_{\min}$, and $\mathcal{C}_{\min}$; alignment identity printing $\cos^2$ agreement; torus coercivity constant with an explicit $\gamma\lambda$ bound and a sampled check; single-axiom falsifiers for symmetry of G, locality, positivity margin, and no-work, each with clear console trip lines; local tomography of G from scalar maps with a two-state cross-check; and the orthogonality identity $\langle v_{\mathrm{irr}}, G^{-1}v_{\mathrm{rev}}\rangle_{H_\rho^{-1}} \approx 0$.
- **00__axiom__diagnostics.py** Runs the metriplectic axiom suite: equality refinement check with mass k_0, nonlocal-closure falsifier sweep, conservative versus non-conservative tripwire, reversible no-work identity in 2D, coarse-grain commutator scaling, and probe identifiability of G, all via console dials with pass or fail verdicts.
- **01__wave-dispersion__probe-fft__suite.py** Tests dispersion and reversibility of the conservative limit through probe-based FFT diagnostics. Reports isotropy, damping, and time-reversal errors.
- **02__metriplectic__equivalence__1d__periodic__kkt.py** Verifies the metriplectic equivalence and inequality in a 1D finite-volume geometry using a sparse KKT solver. Confirms $\mathcal{C}_{\min} = \dot{\sigma}/2$ for exact solutions and the bound $\langle v, \mu\rangle^2 \leq 2\,\mathcal{C}_{\min}\,\dot{\sigma}$ for random admissible directions.
- **03__path-entropy__invariance__metriplectic__batch.py** Runs a high-precision 2D batch test of path-integrated entropy production under mixed reversible and dissipative evolution. Confirms the integrated identity $\int \dot{\sigma}\,dt = \Delta F$ across multiple seeds and reversible amplitudes.
- **04__heat-identity__phase-blind__pinnedDC.py** Evaluates the pure dissipative ($\lambda = 0$) case with exact Fourier semigroup integration and DC mass pinning. Verifies monotonic decay of F and non-negative $\dot{\sigma}$, confirming phase-blind invariance.
- **05__metriplectic__identity__phase-blind__pinnedDC.py** Extends the preceding test to include reversible shifts interleaved with exact heat steps. Both isolated and coupled runs satisfy the integral identity within numerical precision.
- **06__metriplectic__commuting-triangle.py** Combines all preceding channels into a single consistency test for the commuting triangle between reversible, dissipative, and constraint flows. Confirms the metriplectic structure preserves identity exactness under composition.
- **01__dispersion.py** Probes linear dispersion and reversibility in the conservative limit via mode injections and FFT readouts. Reports isotropy of group velocity, damping floor, and time reversal errors across shells.
- **02__kk__resolvent.py** Computes the frequency response $\chi(\omega, k)$ from the linearised KKT resolvent and verifies Kramers-Kronig with a single calibration factor $c_{\mathrm{HT}} = 1.000 \pm 0.005$ for the discrete Hilbert transform.
- **03__entropy__phase__eta.py** Traces the controlled mix $v(\eta) = (1-\eta)v_G + \eta v_J$. Logs the equality dial R and complex modulus M along the path, with R dropping monotonically and M pinned to one within estimator floor.
- **04__diffeo__slice.py** Tests slice covariance under smooth relabellings with full Jacobian weights in both operator and pairings, imposing the mean-zero constraint in the pulled measure. Confirms invariance of R and M to numerical floor on matched subspaces.
- **05__coarsegrain__commutator.py** Measures the Gaussian coarse graining

commutator defect across a decade in filter width ℓ. Recovers the ℓ^2 law and flags breakdown under non smooth filters.

- **06_structure_falsifier.py** Single axiom failure suite. Injects asymmetric metric parts, wrong tangent norms, and pulls G outside divergence, and breaks weighted Liouville. Records the resulting signatures in R, M, and the no-work meter.
- **07_holonomy_loop.py** Evaluates the geometric phase Φ of the complex reader on slow parameter loops. Shows sign flip under loop reversal and area scaling for small rectangles in control space.
- **08_sectoriality_scan.py** Scans the linear spectrum of the dissipative generator. Fits $\Re\,\omega(k) \approx -D_2|k|^2 - D_4|k|^4$ and confirms sectorial resolvent bounds and monotone energy decay.
- **09_em_slice_2d.py** Electrodynamic style slice projection in 2D with $J = c\,\rho^{-1}\varepsilon$. Reads $E = -\nabla\mu$ and $B = c\rho$. Quantifies reversible no-work and detects the anomaly term when $\nabla \cdot (\rho J) \neq 0$.
- **10_optical_metric_poisson.py** Identifies the optical metric $g \propto (\rho G)^{-1}$ on a slice and solves the Poisson type law $\nabla \cdot (\rho G \nabla\mu) = \sigma$. Visualises irreversible ray bending under spatial gradients of ρG.
- **11_maxwell_slice.py** Maxwell style consistency checks for the slice analogy. Compares conservative forms built from (E, B) and documents boundary class effects and constant matrix cases.
- **12_covariance_boost.py** Equal time re slicing and boost covariance test. Recomputes the KKT solve and pairings on matched subspaces and confirms that R and M are invariant up to solver tolerance.
- **13_anomaly_inflow.py** Constructs compensator J' to cancel $\nabla \cdot \big(\rho(J + J')\big)$ and restore the reversible no-work identity. Logs $\dot\sigma_{\text{anom}}$ before and after repair.
- **14_holonomy_quantisation.py** Counts the winding $n = \frac{1}{2\pi}\oint d\,\arg\langle\nabla\phi, \nabla\mu\rangle_C$ in two parameter control space. Shows integer stability under small deformations that avoid branch points.
- **15_optical_gravity_lensing.py** Lensing style readout for irreversible rays using the optical metric picture. Measures deflection angles consistent with index gradients and compares with the $n = (\rho G)^{-1/2}$ correspondence.
- **16_uv_sectoriality.py** Ultraviolet control diagnostics for composite observables. Verifies $|\chi(k)|$ decay at least as $|k|^{-2}$ and confirms no external renormalisation is required within the stated class.
- **17_holonomy_curvature_map.py** Computes complex pairing $Z(a, \theta)$ over a control grid and evaluates Berry curvature via plaquette phases. Confirms smooth nonvanishing reader, trivial Chern index $C = 0$ and clean KKT and Liouville diagnostics on the metriplectic holonomy base case.
- **18_holonomy_sanity_check.py** Global sanity scan of $Z(\phi, \theta)$ over an extended control torus. Verifies $|Z|$ remains well bounded away from zero on 240×240 points, with no candidate defects or nontrivial winding, demonstrating absence of spurious monopoles in natural control families.
- **19_holonomy_coarsegrain_invariance.py** Tests coarse graining invariance of the complex reader and Berry curvature. Compares fine and Gaussian coarse grained states, showing phase of Z and Chern index are preserved while $|Z|$ is rescaled by an almost constant factor, evidencing RG stable holonomy.
- **20_protocol_cost_vs_entropy.py** Compares metriplectic path cost

and Shannon entropy change for competing protocols between the same endpoints. Evaluates H^{-1} metric action and length for linear and wiggled amplitude schedules, demonstrating strong path dependence of cost at fixed ΔS and T.

- **22_holonomy_synthetic_monopole.py** Synthetic testbench for the holonomy machinery on a known phase vortex in control space. Uses the same plaquette and loop algorithms to recover total flux $\sum F \approx 2\pi$, Chern number $C = 1$ and loop winding $n = 1$ for loops encircling the origin and $n \approx 0$ otherwise.
- **23_speedlimit_flat_geodesic_check.py** Two-mode $\mathrm{H}_\rho^{-1}(G)$ speed-limit test in a nearly flat patch of density space. Constructs a family $\rho(x; a_1, a_2)$ on a periodic domain and compares four distinct protocols (linear, wait-then-jump, jump-then-wait, overshoot-then-return) between the same endpoints. For each, solves the KKT equation at midpoints and evaluates the interval cost $C_n = \int \rho G|\partial_x\phi|^2 dx$, total action A, length, and entropy change ΔS. In a flat metric patch the straight-line protocol should approximate the geodesic; the script verifies this numerically, with all compressed or wiggled paths showing strictly larger action.
- **24_speedlimit_curved_geodesic_search.py** Geodesic search in a curved $\mathrm{H}_\rho^{-1}(G(\rho))$ patch induced by a density-dependent mobility $G(\rho) = \exp[\gamma_G(\rho/\rho_0 - 1)]$. Uses the same two-mode density family but now the metric is genuinely position-dependent. Benchmarks a baseline linear protocol against an optimised sine-basis control ansatz with fixed endpoints. A batch random search over control coefficients identifies strictly lower-action paths in the curved metric, demonstrating that the straight line in (a_1, a_2) space is generically not a geodesic once G varies with ρ. Outputs action, length, entropy change, and diagnostic KKT convergence for both protocols, along with plots of $a_j(t)$ and $C(t)$.
- **25_fisher_scalar_gravity_checks.py** End-to-end reproducibility script for the Fisher-scalar sector. Verifies three core identities used in the analysis: (i) the operator identity $L_\rho\phi = -\nabla \cdot (\rho\nabla\phi)$ for $\phi = \log(\rho/\rho_0)$; (ii) the Fisher-Laplacian relation $\Delta\Phi_{\text{eff}} = -(c^2/2)(\Delta\rho/\rho - |\nabla\rho|^2/\rho^2)$; and (iii) the self-sourced radial Helmholtz branch. The script solves the radial ODE, recovers the $n = 1$ analytic profile $\rho(r) = \rho_c \sin(kr)/(kr)$, and compares numerical and analytic radius, mass, and diagnostic compactness $GM/(Rc^2)$. All tests return machine-level agreement, confirming the internal consistency of the scalar sector.

Numerical methods

All integrals are evaluated using uniform-grid rectangle or Simpson quadrature rules consistent with the discrete spectral representation. For dissipative channels, DC modes are pinned exactly in Fourier space to ensure strict mass conservation. Time integration uses exact semigroup updates or second-order Heun/Runge-Kutta schemes where appropriate. Reported iteration counts are printed together with $(\gamma_{\min}, \gamma_{\max})$ and $\rho_{\min}$ to expose conditioning; DC modes are pinned exactly in Fourier space for strict mass conservation. Reversible channels are implemented as shift operators in phase or configuration space and preserve the free energy to machine precision. Poisson and KKT solves use preconditioned conjugate gradients on the mean-zero subspace, with relative residual 10^{-10} in the energy norm; the same discrete gradient and divergence are used in forward and adjoint roles to preserve the ρ-weighted pairing. The mean-zero gauge is enforced by zeroing the DC mode at each solve.

The results are verified across independent random seeds, grid resolutions, and parameter sweeps. All reported quantities are reproducible within standard double-precision floating point tolerance.

Reproducibility

Each script runs independently and produces a single console log summarising the diagnostics. No plots are generated or required. Running all scripts in sequence reproduces the complete numerical verification suite supporting the analytical results of this work. All source files are archived at the GitHub repository above.

G Diagnostics and falsifiers

We certify the axioms and the necessity results by five console dials reproduced by `00_axiom_diagnostics.py`. All evaluations use conservative divergence form, a 2/3 spectral projector on nonlinear operations, subspace consistent pairings for the equality certificate, and exact mass from the zero Fourier mode. Boundary and regularity classes are as in Appendices A-B.

G.1 Equality dial with refinement

We measure the gap

$$\Delta_{\text{eq}} \;\equiv\; 2\,P_{\text{irr}}\,\dot{\sigma} \;-\; \langle v_{\text{irr}}, \mu\rangle^2,$$

which must converge to zero under mesh refinement if the local quadratic metric and the steepest descent direction hold. Typical output:

```
== Equality dial with refinement (1D, P-consistent) ==
N=  512 | ... | gap=-1.527e-02 | ... | PASS
N= 1024 | ... | gap=-7.235e-03 | ... | PASS
N= 2048 | ... | gap=-3.518e-03 | ... | PASS
```

Run constants. Scripts print $\rho_{\min}$, $(\gamma_{\min}, \gamma_{\max})$, measured $\kappa_{\min}$, equality gap, PR, and iteration counts, for example:

```
state: min rho=7.075e-01 | G:[gmin=5.00e-01,gmax=1.40e+00] |

kappa_min=1.00e+00 kappa_min=1.00e+00
```

A second backend confirms the equality on a conservative finite-volume KKT scheme (`02_metriplectic_equivalence_1d_periodic_kkt.py`).

The gap decays like O(dx) and the mass integral is at machine zero, certifying Proposition 2.2 and the cost-entropy equality certificate.

G.2 Nonlocal falsifier

We replace $\nabla\mu$ by a smoothed field before forming j, keep P_{irr} local, and remeasure the equality. The certificate fails by order one margins:

```
== Nonlocal falsifier sweep (1D, P-consistent Pirr) ==
N=4096 | sigma=0.20 | ... | gap=2.205416e-01 | VIOLATES equality
N=4096 | sigma=0.60 | ... | gap=4.926686e-01 | VIOLATES equality
N=4096 | sigma=0.90 | ... | gap=2.274482e-01 | VIOLATES equality
N=4096 | sigma=1.20 | ... | gap=6.519462e-02 | VIOLATES equality
```

Hence local quadratic dissipation is required within scope, matching A3 and the destructive side of Proposition 2.2.

G.3 Conservative vs non conservative tripwire

We contrast a conservative update with a non conservative surrogate. The latter leaks mass:

```
== Conservative vs non conservative tripwire (1D, P-consistent) ==
Integral v_cons dx(k0) = -7.051-18 expected near 0
Integral w_non  dx(k0) = 4.162e-03   non-zero indicates mass leak
```

This guards A1 at the discrete level and rules out false positives from boundary or aliasing artefacts.

G.4 No-work certificate for A6

A nonzero reading arises only if antisymmetry or the weighted Liouville constraint is broken at the current ρ.

```
== Reversible no-work identity (2D) ==
PR = 0.000e+00 | relative = 0.000e+00 | ||v_rev||2 = 4.213e-14 | PASS
```

Dial outcomes and causes.

Outcome	Primary cause at fixed ρ
PR ≈ 0 with constant J and exact mass	Within A6 no-work cone
PR $\neq 0$ under same setup	$J^\top \neq -J$ or $\nabla\cdot(\rho J) \neq 0$

This certifies Proposition 2.3 and the weighted Liouville identity of Appendix B.

G.5 H_ρ^{-1} orthogonality readout

We report the weighted pairing $\langle v_{\rm rev}, \phi_{\rm irr}\rangle_{H_\rho^{-1}}$ where $\mathcal{L}_\rho\phi_{\rm irr} = v_{\rm irr}$:

```
== H^{-1}_rho orthogonality (1D) ==
< v_rev , phi_irr >_{H-1(rho)} = 3.2e-13 | PASS
```

Machine-zero values certify metriplectic orthogonality under Proposition 2.4.

G.6 Coarse grain commutator scaling

We report the normalised commutator

$$\mathrm{rel}(\ell) \;=\; \frac{\|C_\ell(\mathcal{Q}(\rho)) - \mathcal{Q}(C_\ell\rho)\|_2}{\|\mathcal{Q}(\rho)\|_2}, \quad \text{and} \quad \frac{\mathrm{rel}(\ell)}{\ell^2}.$$

For small ℓ the ratio stabilises in a narrow band (grid-independent to leading order), in line with Appendix C:

```
== Coarse-grain commutator scaling (1D, P-consistent, normalised) ==
N=4096 | ell=0.10 | ... | (rel)/ell^2 = 2.500e+00
N=4096 | ell=0.20 | ... | (rel)/ell^2 = 1.602e+00
```

G.7 Probe identifiability

A modest probe set yields a well conditioned Gram matrix for G:

```
== Probe identifiability of G (1D) ==
basis size = 24, min sing = 2.421e+00, max sing = 4.903e+02,
cond(B) = 2.025e+02 identifiability verdict = PASS
```

This supports Proposition 2.1 on recoverability of the quadratic action of G. For separation at a fixed ρ one may take small Fourier probe sets of size $m = 2d+2$ in $d = 1, 2, 3$; identifiability is up to the ellipticity window $(\gamma_{\min}, \gamma_{\max})$.

Run metadata. State health for the run shown: `min rho = 7.075e-01, min G = 5.000e-01`. Script: `00_axiom_diagnostics.py`. All excerpts above are from a single execution of the public archive.

Catalogue of tests used in this section.

(T1) **Baseline conservative suite** (Appendix F): probe-FFT dispersion fits $\omega^2 \sim c^2k^2$, anisotropy ellipse fits under $c_x \neq c_y$, bounded energy drift with leapfrog, and a time-reversal round trip. Purpose: establish that the reversible plumbing behaves as intended before metriplectic checks.

(T2) **Equality and inequality on 1D periodic grids** (Appendix F): face-centred $L_{\rho,G}$ and a sparse KKT solve give (i) equality $\mathcal{C}_{\min} = \dot{\sigma}/2$ on the gradient-flow ray $v_0 = -L_{\rho,G}\mu$ using the exact mean-zero potential, and (ii) the global inequality $\langle v, \mu\rangle^2 \leq 2\,\mathcal{C}_{\min}\dot{\sigma}$ for random admissible $v = -L_{\rho,G}\psi$, reported via the ratio $\mathcal{R}$ and alignment angles.

(T3) **Path-entropy invariance, batch** (Appendix F): event-stop at the first $F(t) = F_{\text{target}}$ shows $\int_0^{t_\star} \dot{\sigma}\,dt = F(0) - F_{\text{target}}$ independent of reversible drift J; multiple amplitudes and two spatial patterns confirm invariance across paths.

(T4) **Heat-only identity and phase-blindness** (Appendix F): exact spectral heat with per-step DC pin verifies $\Delta F = \int \dot{\sigma}\,dt$ with rectangle-rule and midpoint variants, mass is exact by construction, and duplicate runs with distinct labels remain identical (phase-blindness).

(T5) **Reversible stirring plus heat, identity on G-steps** (Appendix F): interleave exact shifts $\rho(x) \mapsto \rho(x - v\,\Delta t)$ with heat; accumulate $\dot{\sigma}$ only on G-steps and confirm $\Delta F = \int \dot{\sigma}\,dt$ to tolerance.

(T6) **Commuting-triangle consistency** (Appendix F): bundles the reversible shifts, dissipative steps, and the constrained KKT solve to check that the instantaneous scalars and integrated identities are insensitive to channel ordering within solver tolerance; includes a small-k curvature oracle at uniform density.

G.8 Quantities computed at a fixed state

Let ρ be a fixed strictly positive density, G be symmetric positive, and F be convex. We evaluate:

(i) **Entropy production** $\dot{\sigma}(\rho) = \int \rho\,(\nabla\mu)^\top G(\nabla\mu)\,dx$.

(ii) **Curvature** $\kappa_{\min}(\rho)$ via the Rayleigh problem (E), implemented on the mean-zero subspace under the H_ρ^{-1} pairing [6, 7, 9].

(iii) **Minimal cost** $\mathcal{C}_{\min}(\rho; v)$ by solving $-L_{\rho,G}\phi = v$ with $L_{\rho,G}\phi \equiv -\nabla\cdot(\rho G\nabla\phi)$, then evaluating the energy form in (E). For the equality case we take $v_0 = -L_{\rho,G}\mu$.

Discretisations share the same gradient and divergence to preserve adjointness for the ρ-weighted inner product. Periodic boxes use spectral derivatives with two-thirds de-aliasing; one-dimensional tests use conservative finite differences that are symmetric under the discrete L_ρ^2 pairing. All KKT potentials are computed and paired on the same grid and mean-zero subspace, so $R = \cos^2\theta_{\rho,G}$

is evaluated as a single Hilbert-space cosine without gauge drift.

G.9 Alignment angle and near-equalities

Given v and its KKT potential ϕ solving $-L_{\rho,G}\phi = v$, define

$$\cos\theta_{\rho,G} \equiv \frac{\langle \nabla\phi, \nabla\mu\rangle_{\rho,G}}{\|\nabla\phi\|_{\rho,G}\,\|\nabla\mu\|_{\rho,G}}, \qquad \langle a, b\rangle_{\rho,G} = \int_\Omega \rho\, a^\top G\, b\, dx.$$

Lemma 3.4 gives the identity $\mathcal{R}(\rho; v) = \cos^2\theta_{\rho,G}$, so near-equalities correspond to small angles between $\nabla\phi$ and $\nabla\mu$ in the (ρ, G) metric. For intuition we also report the Wasserstein-tangent proxy

$$\cos\vartheta \equiv \frac{\langle v,\, v_0\rangle_{H_\rho^{-1}}}{\|v\|_{H_\rho^{-1}}\,\|v_0\|_{H_\rho^{-1}}}, \qquad v_0 \equiv -L_{\rho,G}\mu,$$

which coincides with $\theta_{\rho,G}$ when $G = I$ and is equivalent up to ellipticity constants otherwise. When G is uniformly elliptic, $\gamma_{\min}\|\nabla\psi\|_\rho^2 \le \|\nabla\psi\|_{\rho,G}^2 \le \gamma_{\max}\|\nabla\psi\|_\rho^2$, hence

$$\frac{\gamma_{\min}}{\gamma_{\max}}\cos^2\theta_{\rho,G} \;\le\; \cos^2\vartheta \;\le\; \frac{\gamma_{\max}}{\gamma_{\min}}\cos^2\theta_{\rho,G}.$$

G.10 Protocols and state families

Two state families are used.

- **Synthetic snapshots.** Smooth positive fields are constructed by filtering Gaussian samples in Fourier space and renormalising mass to one, optionally followed by a short relaxation under the dissipative channel to generate representative structure while preserving positivity. Positivity is maintained either by a parametrisation $\rho = \rho_{\min} + e^{\vartheta}$ during transient steps or by clipping at machine epsilon for fixed-state evaluations.
- **Flow snapshots.** Short segments of the metriplectic flow (2.5) with $J = 0$ and constant G produce a sequence of fixed states at which the three scalars are evaluated.

Unless noted, two-dimensional runs use spectral derivatives with two-thirds de-aliasing; one-dimensional referee tests use $N \in \{128, 256, 512, 1024\}$ with conservative finite differences. Scripts print reproducible configuration summaries.

G.11 Headline observations

Across state families and grids we observe:

(O1) **Equality on the gradient-flow ray.** For $v = v_0 = -L_{\rho,G}\mu$ the ratio $\mathcal{R}$ matches one to solver tolerance, confirming Proposition 3.1. See (T2).

(O2) **Global inequality for random tangents.** For random admissible $v = -L_{\rho,G}\psi$, the ratio $\mathcal{R}$ lies below one, with a tight envelope given by $\cos^2\theta_{\rho,G}$ as predicted by Lemma 3.4. See (T2).

(O3) **Curvature coercivity and uniform anchor.** The Rayleigh estimate (3.2) is stable across grids and state families; near uniformity the measured smallest curvature agrees with Proposition 3.5. The small-k oracle at $\rho \equiv$ const is exercised in (T6).

(O4) **Identity under composition.** The instantaneous scalars and integrated identities are insensitive, within solver tolerance, to the ordering of reversible shifts, dissipative steps, and the KKT solve. See (T6).

G.12 Path-entropy invariance under reversible drift

Fix a free energy $F[\rho]$ with chemical potential $\mu = \delta F/\delta\rho$, a symmetric positive G, and an antisymmetric J in the ρ-weighted pairing. Consider

$$\partial_t\rho = \nabla\cdot\left(\rho\,(J\nabla\mu + G\nabla\mu)\right), \qquad \dot\sigma(\rho) = \int_\Omega \rho\,(\nabla\mu)^\top G(\nabla\mu)\,dx.$$

Under periodic or no-flux boundaries and $J^\top = -J$ (pointwise or in L^2_ρ),

$$\frac{d}{dt}F[\rho(t)] = \langle\mu,\,\partial_t\rho\rangle = -\int_\Omega \rho\,(\nabla\mu)^\top G(\nabla\mu)\,dx \quad\Rightarrow\quad \frac{dF}{dt} = -\dot\sigma(\rho).$$

Hence for any trajectory that first hits a target level F_{target},

$$\int_0^{t_\star} \dot\sigma(\rho(t))\,dt = F[\rho(0)] - F_{\text{target}},$$

which is independent of J. In particular, the total dissipative entropy to reach the same F_{target} is path-independent across reversible drifts that are antisymmetric in the stated sense.

> *Remark* (Numerical protocol and result). We implement an event stop at the first crossing $F(t) = F_{\text{target}}$ with linear interpolation of t and $\dot\sigma$. On 192^2 periodic grids with $G = I$, $\lambda = 0$, and reversible fields J of amplitudes $0, 2, 5, 10$ in two spatial modes, batch runs give
>
> $$|S_{\text{total}}(J) - S_{\text{total}}(0)| \in [10^{-13},\, 10^{-5}],$$
>
> with a median near 10^{-6}, while arrival times $t_\star$ do differ across J. For every run, $S_{\text{total}} \approx F(0) - F_{\text{target}}$ to within the reported solver tolerance. See (T3) for the batch harness, (T4) for the heat-only oracle, and (T5) for interleaved reversible-dissipative evolutions with $\dot\sigma$ accumulated only on G-steps.

G.13 Numerical details and tolerances

Rayleigh and Poisson solves terminate at relative residual 10^{-10} unless stated. Spectral derivatives use two-thirds de-aliasing in two dimensions. One-dimensional conservative operators preserve symmetry under the discrete L^2_ρ inner product. Random seeds, grid sizes, and tolerances are printed by each script. Full filenames are listed in Appendix F.

Complementarity and outlook. The reversible bracket and Fisher curvature in the companion paper provide the geometric skeleton; the present analysis supplies the dissipative musculature. Both operate under the same local axioms and diagnostics, and both admit falsifiers that fail once the geometry is altered. Taken together they delineate the minimal reversible-irreversible split consistent with information-geometric curvature, without asserting global unification or uniqueness beyond the stated scope.

These are necessity statements within the axioms; we do not extrapolate beyond the stated function spaces, ellipticity bounds, or boundary class.

H Additional consistency checks

Log-Sobolev curvature and relaxation rate

Setting. We consider the heat flow $\partial_t \rho = \Delta\rho$ on the periodic domain $\mathbb{T}_L$ at fixed $\lambda = 0$. At each time we compute the entropy gap $F(t) - F_\infty$ and the Fisher information $I(t) = \int_\Omega |\partial_x \rho|^2/\rho\, dx$. On the L-periodic box with unit mass and $\lambda = 0$, the minimiser is the uniform density $\bar\rho = 1/L$ and $F_\infty = \int_{\mathbb{T}_L} \bar\rho \log\bar\rho\, dx = -\log L$.

The local log-Sobolev estimator is

$$\kappa_{\text{inst}}(t) = \frac{I(t)}{2(F(t) - F_\infty)}, \qquad \widehat{\kappa} = \min_{t\in[0.05T,T]} \kappa_{\text{inst}}(t).$$

We also fit the late-time exponential decay rate r_{fit} of $F(t) - F_\infty$. *Observation.* For a smooth mixed-mode initial ρ on $\mathbb{T}_{40}$ with $N = 512$ and $dt = 2\times10^{-3}$, we obtain $\widehat{\kappa} \simeq 2.47\times10^{-2}$ and $r_{\text{fit}} \simeq 4.93\times10^{-2}$, satisfying $r_{\text{fit}} \approx 2\widehat{\kappa}$ to numerical precision. *Interpretation.*

This shows within the present discretisation that the empirically measured curvature controls the exponential relaxation of $F(t)$, as predicted by the log-Sobolev bound $F(t) - F_\infty \le (F(0) - F_\infty)e^{-2\kappa t}$. No new assertion is made beyond consistency between the measured curvature and the observed relaxation rate.

Algorithm (metric tomography, up to a scalar). Select m band-separated probes v_i; solve $-L_{\rho,G}\phi_i = v_i$ on the mean-zero subspace; form $H_{ij} = \langle \nabla\phi_i, \nabla\phi_j \rangle_{\rho,G} = \int \rho \nabla\phi_i^\top G \nabla\phi_j \, dx$. With $g = (g_{11}, 2g_{12}, g_{22})$ and $H_{ij} = A_{ij} \cdot g$, least-squares recovers g up to a global scale. We report shellwise condition numbers and uncertainty bands.

Talagrand-type transport inequality

Setting. Along the same heat-flow trajectory as above, we computed the squared Wasserstein distance $W_2^2(t)$ between $\rho(t)$ and the uniform density via the monotone rearrangement map $T(x) = LF_\rho(x)$ with cumulative distribution $F_\rho(x) = \int_0^x \rho(y)\,dy$. Using the curvature estimate $\widehat{\kappa}$ from the previous item, the predicted transport-entropy constant is $C_{\text{pred}} = 1/\widehat{\kappa}$. *Observation.* For all times $t \geq 0.2T$, the inequality

$$W_2^2(t) \ \leq \ C_{\text{pred}}\,(F(t) - F_\infty)$$

is satisfied with negative slack $\max_t[W_2^2 - C_{\text{pred}}(F - F_\infty)] \approx -6.7 \times 10^{-4}$. *Interpretation.* Within the accuracy of the pseudospectral scheme and the rearrangement implementation, the transport-entropy relation holds when the constant is chosen from the independently measured curvature.

This supports that the same κ governs both relaxation and transport in this geometry, without asserting new analytic results. This choice coincides with $\kappa_{\min}(\rho)$ for the uniform anchor used in the main text.

Relaxation spectrum and curvature spectrum

Setting. For single-mode perturbations $\rho(x,0) = \rho_{\text{bar}}(1 + \varepsilon \cos k_{\text{phys}} x)$ with small ε, we evolved the heat flow and measured (i) the exponential decay rate $r_{\text{fit}}(k)$ of the entropy gap and (ii) the curvature estimate $\widehat{\kappa}(k) = \text{median}_t\, I/(2(F - F_\infty))$. The linear theory predicts $r_{\text{theory}} = 2k_{\text{phys}}^2$ and $\kappa(k) = k_{\text{phys}}^2$. *Observation.* For $k = 1 \ldots 5$ on $\mathbb{T}_{40}$, we find relative errors $|r_{\text{fit}} - 2k^2|/(2k^2) \lesssim 10^{-4}$ and $|\widehat{\kappa} - k^2|/k^2 \lesssim 10^{-3}$, limited by late-time round-off. *Interpretation.*

The measured relaxation spectrum coincides with the curvature spectrum to numerical precision, confirming that the instantaneous curvature $\kappa(k)$ accurately encodes the equilibration rate of each Fourier mode. This agreement is a direct consistency check of the theoretical identification between curvature and dissipation in the reversible-dissipative geometry.

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Author's email is contact@nomogenetics.com.

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