Fisher Dirichlet Operator Uniqueness

Universal Information Hydrodynamics: the canonical symmetric mobility on densities

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Context. In the irreversible sector, "dynamics" means a map that turns a thermodynamic gradient into a density drift. Naively this looks underdetermined, since many symmetric operators dissipate entropy. UIH isolates a canonical choice by imposing three structural requirements: (i) mass conservation, (ii) locality (divergence form in the continuum, Markov structure in the discrete), and (iii) the Fisher geometry on densities (Dirichlet power as the primitive object). Under these conditions the symmetric mobility is not a modelling choice. It is fixed.

> **One sentence definition.** The Fisher Dirichlet operator is the unique symmetric, mass conserving density mobility whose quadratic form is the Fisher Dirichlet energy, so it fixes irreversible power and survives coarse graining and GKLS embedding.

1. The primitive is the Dirichlet form

The correct invariant object is the dissipation quadratic form on potentials, not an arbitrary matrix representation. Given a density ρ and a uniformly elliptic symmetric field $G(x)$, define

$$\mathcal{E}_{\rho,G}(\varphi) \;=\; \int_\Omega \rho\,(\nabla\varphi)\cdot G(\nabla\varphi)\,dx, \qquad \mathcal{E}_{\rho,G} \geq 0, \qquad \varphi \equiv \text{const} \Rightarrow \mathcal{E}_{\rho,G} = 0.$$

The associated self-adjoint operator (in the weighted pairing) is the divergence form map

$$L_{\rho,G}\varphi \;=\; -\nabla\cdot(\rho\, G\nabla\varphi), \qquad \mathcal{E}_{\rho,G}(\varphi) = -\langle\varphi,\, L_{\rho,G}\varphi\rangle.$$

Uniqueness here is literal: once the form is fixed, the operator representative is fixed (with the chosen boundary conditions).

2. Discrete state spaces: the Markovian representative

Let Q be a Markov generator with stationary law π (so $Q\pi = 0$, $\sum_i Q_{ij} = 0$). Write a mass conserving perturbation as $\delta p = \mathrm{diag}(\pi)\phi$ with $\sum_i \delta p_i = 0$. The classical Fisher Dirichlet energy is

$$\mathcal{E}_{\mathrm{cl}}(\delta p) = \frac{1}{2}\sum_{i,j} \pi_i w_{ij}\,(\phi_j - \phi_i)^2 \quad \text{with } w_{ij} = Q_{ji}.$$

The corresponding density-sector Fisher operator is the Markovian representative

$$G_{\mathrm{true}} \;:=\; Q\,\mathrm{diag}(\pi), \qquad \mathcal{E}_{\mathrm{cl}}(\delta p) = \langle\phi,\, -G_{\mathrm{true}}\phi\rangle_{\ell^2(\pi)}.$$

This is the sense in which the operator is unique: fixing the density drift Q and stationary weights π fixes the symmetric Dirichlet structure (hence the irreversible power) on the entire mass conserving density subspace. Circulation freedom, when present, lives in the skew channel and does not change $\mathcal{E}_{\mathrm{cl}}$.

3. GKLS embedding: density-sector universality

Let K be a GKLS generator with a full-rank stationary state ρ_{ss}. Place the BKM metric at ρ_{ss} and form the metric adjoint $K^\sharp$. Define the symmetric Fisher part

$$G \;=\; \tfrac{1}{2}\big(K + K^\sharp\big).$$

Restrict to diagonal perturbations in the eigenbasis of ρ_{ss} (population changes only). Then the GKLS Dirichlet energy built from G coincides with the classical Fisher Dirichlet energy of the induced density-sector Markov generator Q:

$$\mathcal{E}_{\mathrm{GKLS}}(\delta p) \;=\; \mathcal{E}_{\mathrm{cl}}(\delta p) \quad \text{for all mass conserving } \delta p.$$

Operationally: coherent dressing can change reversible circulation (the J channel), but it cannot change the canonical symmetric density Dirichlet structure.

4. What would falsify this, cleanly

1. **Non-Dirichlet symmetric mobility:** produce a symmetric local density mobility that conserves mass but whose dissipation quadratic form is not of Fisher Dirichlet type, while still claiming Fisher geometry. That is an internal contradiction: it changes the geometry.
2. **GKLS mismatch:** exhibit a GKLS model with full-rank ρ_{ss} where the BKM symmetric split gives a density Dirichlet energy that differs from the induced Markov Fisher Dirichlet energy for some diagonal δp. That would directly refute density-sector universality.
3. **Coarse graining failure:** show a compatible coarse graining map under which the Fisher Dirichlet form does not descend (or does not contract as required), while the model still claims Fisher-consistent coarse graining.

5. Practical takeaway

The Fisher Dirichlet operator is the canonical "symmetric part" of irreversible density dynamics. It is not an aesthetic choice, and it is not a free parameter. Once you commit to Fisher geometry, locality, and mass conservation, the symmetric mobility on densities is fixed, and any additional freedom is circulation in the skew channel.