Fisher-Kähler Manifold

Complex structure from information plus symplectic form: A Minimal Note

J. R. Dunkley

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Context: why this object keeps appearing. UIH repeatedly meets the same pair: an information metric g controlling distinguishability and dissipation, and a reversible two form ω controlling coherent transport. The Fisher-Kähler manifold is the minimal geometry that fuses these into a single intrinsic structure, so reversible and irreversible motion become two faces of one current.

The Proposal. Given (M, g, ω), form the endomorphism $B := g^{-1}\omega$. Under a mild positivity condition, B canonically induces a complex structure I and a metric h making (M, h, ω, I) Kähler. The key point is that I is not chosen by hand, it is extracted from information geometry.

Core statement (minimal form):
If $B_\rho := g_\rho^{-1}\omega_\rho$ satisfies $-B_\rho^2 \succ 0$, then there is a canonical I_ρ with $I_\rho^2 = -1$ and a canonical metric h_ρ such that Hamiltonian and gradient flows differ by a quarter turn: $X_\Phi = -I\,\nabla_h\Phi$.

1. What it is

A Fisher-Kähler manifold is a state space M equipped with an information metric g and a symplectic form ω such that these data determine a Kähler triple (h, ω, I). "Fisher" means g is the information metric (classical Fisher, or its quantum analogue on faithful states). "Kähler" means there is a compatible complex structure I so the geometry supports both gradient descent and Hamiltonian rotation in one language.

2. Canonical untwisting

At each point $\rho \in M$ define

$$B_\rho := g_\rho^{-1}\omega_\rho.$$

Assume $-B_\rho^2$ is positive definite. Then define

$$S_\rho := \sqrt{-B_\rho^2}, \qquad I_\rho := S_\rho^{-1}B_\rho, \qquad h_\rho(X,Y) := g_\rho(S_\rho X, Y).$$

This gives $I_\rho^2 = -1$. When I is integrable and $\omega(\cdot,\cdot) = h(\cdot, I\cdot)$, the triple (h, ω, I) is Kähler. The construction is canonical: once g and ω are fixed, I is forced.

3. Gradients versus Hamiltonians

For any functional $\Phi : M \to \mathbb{R}$, define the h gradient $\nabla_h\Phi$ and the Hamiltonian field X_Φ by

$$h(\nabla_h\Phi, X) = d\Phi(X), \qquad \omega(X_\Phi, X) = d\Phi(X).$$

On a Fisher-Kähler manifold these are related by

$$X_\Phi = -I\,\nabla_h\Phi.$$

Interpretation: irreversible descent and reversible transport are the same drive, expressed in two conjugate quadratures fixed by I.

4. Why it is the right home for $K = G + J$

UIH packages dynamics into a single generator K with symmetric part G (dissipative, contractive) and antisymmetric part J (reversible, Liouville type). The Fisher-Kähler manifold is the geometric avatar of that decomposition. It tells you when a given reduced description genuinely supports an intrinsic complex rotation between the G and J components, rather than a coordinate artefact.

5. What to compute

Given data that lets you estimate g and ω (or a reduced generator split into symmetric and antisymmetric parts), you can compute $B = g^{-1}\omega$, check the condition $-B^2 \succ 0$, then build S, I, and h. Two practical diagnostics follow immediately: how close I^2 is to -1 numerically, and how well ω matches the compatibility form $h(\cdot, I\cdot)$. These are direct, machine checkable tests of "Kähler quality" of the inferred information dynamics.

6. UIH viewpoint

The Fisher-Kähler manifold is the clean geometric statement: one current, two quadratures. It clarifies how the Fisher sector, generator splitting, tomography based reconstruction, and coarse graining all fit into a single invariant object that can be probed and falsified.