Net chiral index on a minimal internal Fisher Kähler fibre

Open Technical Note

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Contents

1 Purpose

The aim of this open technical note is to record a rigid, purely topological statement for a minimal internal fibre supporting colour and weak symmetries, together with a natural choice of bundle data. The output is a closed form polynomial for a twisted spin^c Dirac index on a compact complex threefold Y, with discrete dependence on an SU(3) instanton number. The note is deliberately narrow. It isolates one index computation tied to one minimal internal bundle choice, and it keeps all twisting globally well defined.

A second, explicitly marked aim is to sharpen the open dynamical continuation. Theorem 5.2 below shows that, in the globally honest hypercharge packaging, literal replication by tuning global line bundle flux does not occur. If replication and mixing occur in this framework, they must enter through multiplicity of localised sectors above a vacuum label, with mixing controlled by the spectral data of $\mathcal{K}$ and its effective slow subspaces. Section 8 records a canonical three well mechanism native to CP^2 and states the corresponding qutrit slow sector target. In the updated version of this note, Section 8 also records disciplined computational diagnostics for that target on simplex discretisations of the moment polytope, to pin down the correct barrier height and scaling regime for a subsequent analytic theorem.

2 Internal fibre and bundle data

2.1 The fibre

Let

$$Y = \mathrm{CP}^2 \times \mathrm{CP}^1. \tag{2.1}$$

Write $h_3 \in H^2(\mathrm{CP}^2; \mathbb{Z})$ for the hyperplane class on CP^2, and $h_2 \in H^2(\mathrm{CP}^1; \mathbb{Z})$ for the hyperplane class on CP^1. We identify these with their pullbacks to Y. The cohomology ring is generated by h_3, h_2 with relations

$$h_3^3 = 0, \qquad h_2^2 = 0, \tag{2.2}$$

and normalisation

$$\int_Y h_3^2 h_2 = 1. \tag{2.3}$$

The total Todd class factorises,

$$\mathrm{Td}(TY) = \mathrm{Td}(T\mathrm{CP}^2)\,\mathrm{Td}(T\mathrm{CP}^1) = \big(1 + \tfrac{3}{2}h_3 + h_3^2\big)\big(1 + h_2\big), \tag{2.4}$$

so in particular the top degree term is $(\mathrm{Td}(TY))_6 = h_3^2 h_2$. Since CP^2 is not spin, Y is not spin. We work with the canonical spin^c structure associated to the complex structure, for which the index is computed by the Todd class $\mathrm{Td}(TY)$.

2.2 Gauge bundles

We specify an SU(3) bundle $E_3 \to Y$ pulled back from CP^2, with

$$c_1(E_3) = 0, \qquad c_2(E_3) = n_3\, h_3^2, \qquad n_3 \in \mathbb{Z}, \tag{2.5}$$

and we impose $c_3(E_3) = 0$ in the minimal branch considered in the first part, since it does not enter a degree six computation on Y. The corresponding Chern character truncates as

$$\mathrm{ch}(E_3) = 3 - n_3\, h_3^2, \tag{2.6}$$

since higher degree terms do not contribute to a degree six integral on Y under the assumptions above.

We take an SU(2) bundle $E_2 \to Y$ pulled back from CP^1. In the minimal topology branch we treat E_2 as topologically trivial, so

$$\mathrm{ch}(E_2) = 2. \tag{2.7}$$

2.3 Hypercharge twisting as an honest line bundle

To avoid fractional tensor powers, we introduce a primitive line bundle $L_0 \to Y$ with

$$c_1(L_0) = a\, h_3 + b\, h_2, \qquad a, b \in \mathbb{Z}. \tag{2.8}$$

We encode the Standard Model hypercharges by integer exponents $k_R = 6Y_R \in \mathbb{Z}$ and define the associated line bundles as $L_0^{k_R}$. Concretely,

$$k_q = 1, \qquad k_u = 4, \qquad k_d = -2, \qquad k_\ell = -3, \qquad k_e = -6. \tag{2.9}$$

This packaging is consistent with the usual statement that the faithful gauge group is a discrete quotient in which hypercharge is quantised in units of $1/6$. We will not need any finer global information here. For any $k \in \mathbb{Z}$,

$$\mathrm{ch}(L_0^k) = \exp\big(k(ah_3 + bh_2)\big), \tag{2.10}$$

understood as a truncated series in cohomology using $h_3^3 = 0$ and $h_2^2 = 0$.

2.4 Standard Model multiplet bundles

Define the holomorphic bundles corresponding to one Standard Model generation as

$$Q_L = E_3 \otimes E_2 \otimes L_0^{k_q}, \qquad U_R = E_3 \otimes L_0^{k_u}, \qquad D_R = E_3 \otimes L_0^{k_d}, \tag{2.11}$$

$$L_L = E_2 \otimes L_0^{k_\ell}, \qquad E_R = L_0^{k_e}. \tag{2.12}$$

Let

$$E_{\mathrm{SM}} := Q_L \oplus U_R \oplus D_R \oplus L_L \oplus E_R. \tag{2.13}$$

We will study the twisted spin^c Dirac operator $D^Y_{E_{\mathrm{SM}}}$ and its index.

3 Index computation

3.1 Definition

We define the net Standard Model chiral index in this internal sector by

$$I_{\mathrm{SM}}(a,b,n_3) := \mathrm{index}\big(D^Y_{E_{\mathrm{SM}}}\big) = \int_Y \mathrm{ch}(E_{\mathrm{SM}})\,\mathrm{Td}(TY). \tag{3.1}$$

This is an integer and depends only on the topological data (a, b, n_3) under the minimal branch assumptions above.

3.2 Closed form polynomial

Proposition 3.1 . *For the bundle choice above one has the closed form*

$$I_{\mathrm{SM}}(a,b,n_3) = -48a^2b + 60a^2 + 180ab - 4bn_3 - 4n_3 + 15. \tag{3.2}$$

Proof. One multiplies the truncated Chern characters $\mathrm{ch}(E_3) = 3 - n_3h_3^2$, $\mathrm{ch}(E_2) = 2$, and $\mathrm{ch}(L_0^k) = \exp(k(ah_3 + bh_2))$, forms $\mathrm{ch}(E_{\mathrm{SM}})$ by additivity, then multiplies by $\mathrm{Td}(TY)$ from Eq. (2.4). The integral over Y extracts the coefficient of $h_3^2h_2$ using $\int_Y h_3^2h_2 = 1$. □

3.3 Per multiplet indices

Since $D^Y_{E_{\mathrm{SM}}}$ is block diagonal in the multiplet decomposition, its index is the sum of the indices of the twisted operators for each bundle. The individual indices are also rigid polynomials.

Lemma 3.2 . *The multiplet indices are*

$$I_Q(a,b,n_3) := \mathrm{index}\big(D^Y_{Q_L}\big) = 3a^2b + 3a^2 + 9ab + 9a + 6b - 2n_3(b+1) + 6, \tag{3.3}$$

$$I_U(a,b,n_3) := \mathrm{index}\big(D^Y_{U_R}\big) = 96a^2b + 24a^2 + 72ab + 18a + 12b - n_3(4b+1) + 3, \tag{3.4}$$

$$I_D(a,b,n_3) := \mathrm{index}\big(D^Y_{D_R}\big) = -12a^2b + 6a^2 + 18ab - 9a - 6b + n_3(2b-1) + 3, \tag{3.5}$$

$$I_L(a,b) := \mathrm{index}\big(D^Y_{L_L}\big) = -27a^2b + 9a^2 + 27ab - 9a - 6b + 2, \tag{3.6}$$

$$I_E(a,b) := \mathrm{index}\big(D^Y_{E_R}\big) = -108a^2b + 18a^2 + 54ab - 9a - 6b + 1, \tag{3.7}$$

and satisfy

$$I_{\mathrm{SM}} = I_Q + I_U + I_D + I_L + I_E. \tag{3.8}$$

Proof. Same computation as in the proposition, but applied to each summand bundle separately, using multiplicativity of ch under tensor product and additivity under direct sum. □

4 Minimal branch consequences

4.1 The $(a, b) = (0, 0)$ branch

A particularly transparent regime is the minimal hypercharge branch

$$a = 0, \qquad b = 0, \tag{4.1}$$

in which the hypercharge twisting line bundle is topologically trivial, $L_0 \simeq \mathcal{O}_Y$, and all dependence on (a, b) disappears. In this branch the only remaining discrete parameter in the present truncation is the colour instanton number n_3 through $c_2(E_3) = n_3 h_3^2$.

Specialising the multiplet polynomials of Lemma 3.2 yields

$$I_Q(0, 0, n_3) = 6 - 2n_3, \tag{4.2}$$
$$I_U(0, 0, n_3) = 3 - n_3, \tag{4.3}$$
$$I_D(0, 0, n_3) = 3 - n_3, \tag{4.4}$$
$$I_L(0, 0) = 2, \tag{4.5}$$
$$I_E(0, 0) = 1, \tag{4.6}$$

and therefore

$$I_{\mathrm{SM}}(0, 0, n_3) = -4n_3 + 15. \tag{4.7}$$

This is the simplest statement of the note: within the globally honest hypercharge packaging, and within the minimal topology branch for weak isospin, the net chiral index depends on the colour instanton number only through the colour charged multiplets, while the lepton indices remain fixed.

> *Remark.* The fixed values $I_L = 2$ and $I_E = 1$ in the minimal branch are not a claim of three family replication. They are the indices of the chosen internal twisting for a single fixed vacuum label. In particular, the value 15 at $n_3 = 0$ should be read only as the output of the chosen internal bundle data and the chosen convention for what is included in E_{SM}.

4.2 Dependence on n_3 is confined to colour charged multiplets

The full polynomial makes the confinement of n_3 dependence explicit. From Lemma 3.2 one has

$$I_L(a, b) \text{ and } I_E(a, b) \text{ are independent of } n_3, \tag{4.8}$$

whereas

$$I_Q(a, b, n_3),\ I_U(a, b, n_3),\ I_D(a, b, n_3) \tag{4.9}$$

carry affine dependence on n_3 with coefficients fixed by the ranks and hypercharge exponents. This is the minimal version of the claim that the colour instanton number controls only the colour sector contribution to the net chiral index within this internal fibre choice.

5 No replication by global hypercharge flux

The preceding index formulae permit a direct obstruction statement. If one attempts to realise N identical lepton families purely by tuning the global line bundle flux (a, b), one would require the lepton indices to scale as

$$I_L(a, b) = 2N, \qquad I_E(a, b) = N, \tag{5.1}$$

with $N \geq 2$ an integer.

> **Theorem 5.1 .** *There is no integer choice $(a, b) \in \mathbb{Z}^2$ and integer $N \geq 2$ such that*
>
> $$I_L(a, b) = 2N, \qquad I_E(a, b) = N. \tag{5.2}$$
>
> *The only solution is the trivial one $N = 1$ with $(a, b) = (0, 0)$.*

Proof. Assume $I_L(a, b) = 2N$ and $I_E(a, b) = N$. Eliminating N gives

$$2I_E(a, b) - I_L(a, b) = 0. \tag{5.3}$$

Substituting the explicit polynomials from Lemma 3.2 yields the Diophantine equation

$$63a^2 b - 27ab + 2b = 9a^2 - 3a. \tag{5.4}$$

Equivalently,

$$b\big(63a^2 - 27a + 2\big) = 3a(3a - 1). \tag{5.5}$$

For integer a, the quadratic

$$63a^2 - 27a + 2 \tag{5.6}$$

is strictly positive. Moreover, for any nonzero integer a one has the strict inequality

$$0 < 3a(3a - 1) < 63a^2 - 27a + 2, \tag{5.7}$$

so the ratio

$$b = \frac{3a(3a - 1)}{63a^2 - 27a + 2} \tag{5.8}$$

is a proper fraction and cannot be an integer. Hence $a \neq 0$ is impossible. If $a = 0$ then the equation forces $b = 0$. Substituting back gives $I_L(0, 0) = 2$ and $I_E(0, 0) = 1$, so $N = 1$. This is the only solution. □

> *Remark.* The content of this theorem is deliberately narrow. It does not say that family replication is impossible in the framework. It says that replication cannot be implemented as literal multiplication of the lepton indices by tuning a global hypercharge flux within the integer packaged line bundle L_0. Any three family mechanism must therefore be dynamical and localised, with replication living in multiplicity of localised sectors above a fixed vacuum label, rather than in global

line bundle exponents.

6 Holomorphic representatives and an extended topology branch

6.1 A holomorphic $\mathrm{SU}(3)$ bundle on CP^2 with $c_2 = n_3 h_3^2$

The index computation above depends only on the Chern character. One may nevertheless record an explicit holomorphic representative of the colour bundle on CP^2 with the required second Chern class, to underline that the topological input is not empty.

Proposition 6.1 . *For each $n \in \mathbb{N}$ there exists a rank three holomorphic vector bundle $E(n) \to \mathrm{CP}^2$ defined as the cohomology of the monad*

$$0 \longrightarrow \mathcal{O}_{\mathrm{CP}^2}(-1)^{\oplus n} \longrightarrow \mathcal{O}_{\mathrm{CP}^2}^{\oplus(2n+3)} \longrightarrow \mathcal{O}_{\mathrm{CP}^2}(1)^{\oplus n} \longrightarrow 0, \tag{6.1}$$

such that

$$\mathrm{ch}\big(E(n)\big) = 3 - n\, h_3^2, \tag{6.2}$$

hence $c_1\big(E(n)\big) = 0$ and $c_2\big(E(n)\big) = n\, h_3^2$.

Proof. For a monad $A \to B \to C$ with cohomology bundle $E = \ker(B \to C)/\mathrm{im}(A \to B)$, one has

$$\mathrm{ch}(E) = \mathrm{ch}(B) - \mathrm{ch}(A) - \mathrm{ch}(C). \tag{6.3}$$

Here $\mathrm{ch}(\mathcal{O}(1)) = \mathrm{e}^{h_3}$ and $\mathrm{ch}(\mathcal{O}(-1)) = \mathrm{e}^{-h_3}$ on CP^2, so

$$\mathrm{ch}(A) + \mathrm{ch}(C) = n\mathrm{e}^{-h_3} + n\mathrm{e}^{h_3} = 2n + nh_3^2, \tag{6.4}$$

using $h_3^3 = 0$. Since $\mathrm{ch}(B) = 2n + 3$, the stated formula follows. □

Pulling back $E(n)$ along the projection $\mathrm{CP}^2 \times \mathrm{CP}^1 \to \mathrm{CP}^2$ gives a holomorphic $\mathrm{SU}(3)$ bundle on Y with $c_2 = nh_3^2$ as assumed above.

6.2 Remarks on possible extensions

The present index computation is intentionally minimal. Additional discrete labels can be introduced by relaxing the minimal branch assumptions, for example by allowing further topological data in the weak and hypercharge sectors, or by allowing additional internal bundles beyond E_{SM}.

Any such extension should respect two constraints if the note is to remain a clean theorem statement. First, the hypercharge twisting should remain globally honest, avoiding fractional tensor powers. Second, the extension should not weaken the obstruction of Theorem 5.1 by allowing replication to be encoded purely in a change of global flux. If replication and mixing are to be claimed, they should be claimed as consequences of dynamical localisation and slow sector reduction, not as consequences of reparameterising global topological integers.

7 Interpretation and scope boundary

The index polynomial of Proposition 3.1 is a rigid topological output for a fixed internal fibre and a fixed internal bundle choice. The dependence on n_3 is discrete and controlled, and in the minimal $(a, b) = (0, 0)$ branch the lepton indices are frozen while the colour sector varies with n_3.

Theorem 5.1 is the key scope delimiter. It rules out an easy but misleading route to "three families", namely multiplying lepton indices by tuning a global hypercharge flux. If replication and mixing are to be discussed in this setting without overclaiming, then replication must be a dynamical multiplicity phenomenon above a fixed vacuum label, and mixing must be read as spectral and overlap data of an effective slow sector of the generator $\mathcal{K}$.

The remaining sections formalise a canonical three well mechanism that is native to CP^2 and compatible with this scope. The mechanism does not touch the global index polynomial. It acts within a vacuum sector by creating three long lived localised packets whose slow exchange dynamics closes to a 3×3 operator

$$\frac{\mathrm{d}p}{\mathrm{d}t} = K_{\mathrm{slow}}\, p, \qquad p \in \mathbb{R}^3, \qquad 1^\top K_{\mathrm{slow}} = 0, \tag{7.1}$$

and one then interprets replication and mixing, if present, through the localisation structure and the spectral data of K_{slow}.

8 A canonical three well Fisher sector on CP^2

8.1 Moment simplex and fixed points

The complex projective plane CP^2 is toric. Under the standard Hamiltonian T^2 action, the moment map

$$\mu : \mathrm{CP}^2 \longrightarrow \Delta_2 \tag{8.1}$$

has image the standard simplex

$$\Delta_2 := \left\{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3_{\geq 0} :\ \mu_1 + \mu_2 + \mu_3 = 1\right\}. \tag{8.2}$$

The three torus fixed points map to the three vertices of Δ_2, which we denote by

$$v_1 = (1, 0, 0), \qquad v_2 = (0, 1, 0), \qquad v_3 = (0, 0, 1). \tag{8.3}$$

In what follows we treat μ as a coarse coordinate on CP^2 and we build a selection mechanism that is expressed purely in terms of μ.

8.2 A canonical symmetric three well potential

Define the symmetric polynomial

$$V(\mu) := \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, \qquad \mu \in \Delta_2. \tag{8.4}$$

This potential is nonnegative on Δ_2 and vanishes precisely at the vertices. It has a single interior critical point at the barycentre $\mu_* = (1/3, 1/3, 1/3)$ and admits an elementary barrier height.

Lemma 8.1 . *Along each edge of* Δ_2*, the restriction of* V *has a unique maximum of height* $1/4$ *attained at the midpoint of the edge. In particular, the minimal barrier height separating any vertex from any other vertex through the simplex is*

$$\Delta V = \frac{1}{4}. \tag{8.5}$$

Proof. Consider the edge $\mu_3 = 0$ with $\mu_1 + \mu_2 = 1$ and $\mu_1, \mu_2 \geq 0$. Then $V(\mu) = \mu_1\mu_2$ on that edge, whose maximum on $[0, 1]$ occurs at $\mu_1 = \mu_2 = 1/2$ with value $1/4$. The same computation applies to the other edges by symmetry. □

8.3 Fisher regularised selection and concentration

Let ρ be a probability density on CP^2 with respect to the standard volume form dvol, and let $\mu(y) \in \Delta_2$ denote the moment map coordinate. For $\varepsilon > 0$ define the Fisher regularised selection functional

$$F_\varepsilon[\rho] := \varepsilon\, \mathcal{I}[\rho] + \int_{\mathrm{CP}^2} V\big(\mu(y)\big)\, \rho(y)\, \mathrm{dvol}(y), \tag{8.6}$$

where $\mathcal{I}[\rho]$ is the Fisher information in the underlying Kähler metric. We impose a moment constraint

$$\int_{\mathrm{CP}^2} \mu_i(y)\, \rho(y)\, \mathrm{dvol}(y) = m_i, \qquad i = 1, 2, 3, \tag{8.7}$$

for some $m = (m_1, m_2, m_3) \in \Delta_2$.

Proposition 8.2 (*Selection limit on* CP^2). *Fix an interior moment constraint* $m \in \mathrm{int}(\Delta_2)$*. Let* ρ_ε *be a minimiser of* F_ε *subject to Eq. (8.7). As* $\varepsilon \to 0$*, any weak limit point of the pushforward measures* $\mu_\#(\rho_\varepsilon\, \mathrm{dvol})$ *is supported on the vertices* $\{v_1, v_2, v_3\}$*.*

If moreover $m = \mu_* = (1/3, 1/3, 1/3)$*, then the limiting weights are forced to be equal:*

$$\mu_\#(\rho_\varepsilon\, \mathrm{dvol}) \rightharpoonup \frac{1}{3}\delta_{v_1} + \frac{1}{3}\delta_{v_2} + \frac{1}{3}\delta_{v_3}. \tag{8.8}$$

Proof sketch. The potential term in Eq. (8.6) is minimised by measures concentrating where V is minimal, namely at the vertices of Δ_2. The Fisher term is a coercive regulariser which prevents pathological oscillation and provides tightness. In the limit $\varepsilon \to 0$ the minimisers converge, in the sense of Γ convergence, to minimisers of the potential term under the moment constraint.

An interior constraint $m \in \mathrm{int}(\Delta_2)$ cannot be realised by a measure supported on fewer than three vertices, since any convex combination of two distinct vertices lies on an edge. Hence the limit support must include all three vertices. If $m = \mu_*$, symmetry forces equal weights. □

8.4 Metastable qutrit slow sector target

The selection limit suggests that, for small ε, the relevant degrees of freedom are the three packet weights near the fixed points. Let $U_i \subset \mathrm{CP}^2$ be disjoint neighbourhoods of the three fixed points with $\mu(U_i)$ contained in small neighbourhoods of v_i. For an evolving density $\rho(t, y)$ define the packet masses

$$p_i(t) := \int_{U_i} \rho(t, y)\, \mathrm{dvol}(y), \qquad i = 1, 2, 3, \tag{8.9}$$

so that $p(t) \in \mathbb{R}^3$ is a coarse description of the internal state once localisation has formed.

The analytic goal is then a metastable reduction theorem: the full infinite dimensional evolution generated by $\mathcal{K}$ admits, for small ε, a three dimensional slow manifold on which the packet masses close to a linear qutrit dynamics.

> *Remark* (Target statement for a second note). One seeks an existence plus spectral gap theorem of the following form. For ε sufficiently small, there exist three slow modes separated by a gap from the fast spectrum, and there exists a reduced operator $K_{\mathrm{slow}}(\varepsilon)$ such that, after an initial transient, the packet masses satisfy
>
> $$\frac{\mathrm{d}p}{\mathrm{d}t} = K_{\mathrm{slow}}(\varepsilon)\, p + \text{higher order corrections}, \tag{8.10}$$
>
> with $1^\top K_{\mathrm{slow}} = 0$ and with a decomposition
>
> $$K_{\mathrm{slow}} = G_{\mathrm{slow}} + J_{\mathrm{slow}} \tag{8.11}$$
>
> that is the induced symmetric and antisymmetric split under the induced inner product on the slow manifold.
>
> A sharp quantitative target is that the slow decay rates are exponentially small with barrier $\Delta V = 1/4$ from Lemma 8.1, in the form
>
> $$-\Re\lambda_{\mathrm{slow}}(\varepsilon) \asymp \exp\!\Big(-\frac{\Delta V}{\varepsilon}\Big), \qquad \Delta V = \frac{1}{4}, \tag{8.12}$$
>
> up to subexponential prefactors determined by local curvature and mobility data.

8.5 Disciplined simplex diagnostics for the slow sector

To pin down the correct barrier and scaling regime, one can work on a finite simplex lattice approximation of Δ_2. Fix an integer $N \geq 1$ and define the lattice

$$\Delta_2^{(N)} := \Big\{\mu \in \Delta_2 : \ \mu_i \in \frac{1}{N}\mathbb{Z}\Big\}, \tag{8.13}$$

with nearest neighbour adjacency given by the standard triangular lattice moves. Let V be the discrete restriction of **??** to $\Delta_2^{(N)}$, and define the Gibbs weights

$$\pi(\mu) := \frac{\exp\big(-V(\mu)/\varepsilon\big)}{\sum_{\nu\in\Delta_2^{(N)}} \exp\big(-V(\nu)/\varepsilon\big)}. \tag{8.14}$$

A reversible conductance on each undirected edge (μ, ν) is

$$w_{\mu\nu} := \exp\Big(-\frac{V(\mu) + V(\nu)}{2\varepsilon}\Big), \tag{8.15}$$

which is symmetric. To build a nonreversible circulation consistent with stationarity, one chooses an oriented boundary cycle and an antisymmetric conductance $q_{\mu\nu} = -q_{\nu\mu}$ supported only on boundary edges, with constant magnitude $f > 0$ and with nodewise divergence free condition

$$\sum_{\nu:\ (\mu,\nu)\ \mathrm{edge}} q_{\mu\nu} = 0 \qquad \text{for all } \mu \in \Delta_2^{(N)}. \tag{8.16}$$

Then the total conductance is

$$c_{\mu\nu} := w_{\mu\nu} + q_{\mu\nu}, \tag{8.17}$$

and positivity is ensured by choosing f below the minimum of $w_{\mu\nu}$ on boundary edges. The column generator Q acting on probability vectors p is defined by

$$Q_{\mu\nu} := \begin{cases} c_{\mu\nu}/\pi(\nu), & \mu \neq \nu \text{ and } (\mu,\nu) \text{ is an edge,} \\ -\sum_{\eta\neq\nu} Q_{\eta\nu}, & \mu = \nu, \\ 0, & \text{otherwise.} \end{cases} \tag{8.18}$$

By construction one has

$$Q\pi = 0, \tag{8.19}$$

and the antisymmetric part of Q is a finite dimensional analogue of J that preserves the stationary state.

The diagnostic task is then to compute the leading spectrum of Q and verify that the slow eigenvalues are exponentially small in ε with barrier $\Delta V = 1/4$, and that the slow subspace is three dimensional and well separated from the fast spectrum. An effective 3×3 operator K_{slow} can be extracted by projecting corner indicators into the slow subspace under the π weighted inner product and then Galerkin reducing Q onto that projected basis.

Remark (Observed scaling and gap in the discrete diagnostic). In the simplex diagnostic described above, with a fixed circulation strength and with ε swept through the small regime, the leading nonzero slow decay rate obeys

$$\log\big(-\Re\lambda_{\mathrm{slow}}(\varepsilon)\big) \approx -\frac{\Delta V_{\mathrm{fit}}}{\varepsilon} + \mathrm{const}, \tag{8.20}$$

with $\Delta V_{\mathrm{fit}} \simeq 0.252$, consistent with $\Delta V = 1/4$ from Lemma 8.1. In the same regime the next eigenvalue is order one negative, giving a very large timescale separation. This provides a sharp numerical target for the analytic metastable reduction theorem referenced in Remark 8.2.

8.6 Extracting an explicit 3×3 generator from the simplex diagnostic

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We make the qutrit reduction in Remark 8.2 concrete in the discrete setting. Fix three disjoint corner neighbourhoods in the simplex lattice, for example

$$U_1^{(r)} := \{\mu \in \Delta_2^{(N)} : \mu_1 \geq 1-r\}, \qquad U_2^{(r)} := \{\mu \in \Delta_2^{(N)} : \mu_2 \geq 1-r\}, \qquad U_3^{(r)} := \{\mu \in \Delta_2^{(N)} : \mu_3 \geq 1-r\}, \tag{8.21}$$

for some small $r \in (0,1)$. Let χ_i be the normalised indicator vectors of these sets,

$$\chi_i(\mu) := \frac{\mathbf{1}_{U_i^{(r)}}(\mu)}{\|\mathbf{1}_{U_i^{(r)}}\|_2}, \qquad i = 1,2,3, \tag{8.22}$$

viewed as vectors in $\mathbb{R}^{|\Delta_2^{(N)}|}$.

Let (λ_k, v_k) denote the leading right eigenpairs of Q (sorted by real part, with $\lambda_0 = 0$). In the metastable regime one observes a slow triple consisting of the stationary mode and the slow complex conjugate pair. Denote by

$$V_{\mathrm{slow}} := \mathrm{span}\{v_0, \Re v_1, \Im v_1\} \tag{8.23}$$

the associated real three dimensional slow subspace. Let $\langle x, y\rangle_\pi := \sum_\mu x(\mu)y(\mu)\pi(\mu)$ be the π weighted inner product and P_{slow} the π orthogonal projector onto V_{slow}. Define the projected corner basis

$$u_i := P_{\mathrm{slow}}\chi_i, \qquad i = 1,2,3, \tag{8.24}$$

and set $U := [u_1\ u_2\ u_3]$ as a matrix with columns u_i. One then defines a reduced generator by Galerkin projection in the π inner product:

$$K_{\mathrm{slow}} := (U^\top \Pi U)^{-1}\, U^\top \Pi Q U, \qquad \Pi := \mathrm{diag}(\pi), \tag{8.25}$$

followed by enforcing exact column conservation in the reduced model by replacing K_{slow} with

$$K_{\mathrm{slow}} \leftarrow K_{\mathrm{slow}} - \mathrm{diag}\big(1^\top K_{\mathrm{slow}}\big). \tag{8.26}$$

This produces a bona fide 3×3 column generator on the packet masses.

A natural reduced stationary vector is the induced weight of each basis column under π,

$$\Pi_{\mathrm{slow},i} := \langle 1, u_i\rangle_\pi, \qquad i = 1,2,3, \tag{8.27}$$

normalised so that $\sum_i \Pi_{\mathrm{slow},i} = 1$.

8.7 Weighted $G_{\rm slow} + J_{\rm slow}$ split in the reduced model

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Once $\Pi_{\rm slow}$ is fixed, one defines the adjoint of $K_{\rm slow}$ with respect to the weighted inner product on $\mathbb{R}^3$,

$$\langle x, y\rangle_{\Pi_{\rm slow}} := x^\top \operatorname{diag}(\Pi_{\rm slow})\, y. \tag{8.28}$$

The corresponding adjoint is

$$K^*_{\rm slow} := \operatorname{diag}(\Pi_{\rm slow})^{-1} K^\top_{\rm slow} \operatorname{diag}(\Pi_{\rm slow}), \tag{8.29}$$

and the induced symmetric and antisymmetric parts are

$$G_{\rm slow} := \frac{1}{2}\big(K_{\rm slow} + K^*_{\rm slow}\big), \qquad J_{\rm slow} := \frac{1}{2}\big(K_{\rm slow} - K^*_{\rm slow}\big). \tag{8.30}$$

This is the finite dimensional analogue of the $\mathcal{K} = G + J$ split restricted to the slow manifold.

Remark (A fully explicit qutrit instance). For one representative parameter choice in the simplex diagnostic, taking $N = 60$, $\varepsilon = 0.02$, a fixed boundary circulation below the positivity threshold, and a symmetric potential ($\delta = 0$), the extracted reduced generator has the form

$$K_{\rm slow} \approx \begin{pmatrix} -1.2687\times10^{-5} & 7.4107\times10^{-7} & 1.1946\times10^{-5} \\ 1.1946\times10^{-5} & -1.2687\times10^{-5} & 7.4107\times10^{-7} \\ 7.4107\times10^{-7} & 1.1946\times10^{-5} & -1.2687\times10^{-5} \end{pmatrix}, \tag{8.31}$$

with induced stationary weights $\Pi_{\rm slow} = (1/3, 1/3, 1/3)$ by symmetry.

In that case the weighted decomposition Eq. (8.30) gives

$$\begin{aligned} G_{\rm slow} &\approx \begin{pmatrix} -1.2687\times10^{-5} & 6.343535\times10^{-6} & 6.343535\times10^{-6} \\ 6.343535\times10^{-6} & -1.2687\times10^{-5} & 6.343535\times10^{-6} \\ 6.343535\times10^{-6} & 6.343535\times10^{-6} & -1.2687\times10^{-5} \end{pmatrix}, \\ J_{\rm slow} &\approx \begin{pmatrix} 0 & -5.602465\times10^{-6} & 5.602465\times10^{-6} \\ 5.602465\times10^{-6} & 0 & -5.602465\times10^{-6} \\ -5.602465\times10^{-6} & 5.602465\times10^{-6} & 0 \end{pmatrix}. \end{aligned} \tag{8.32}$$

This is a concrete $G + J$ qutrit slow sector in the precise sense of Remark 8.2. It does not prove the continuum theorem, but it pins down the correct symmetry class and the correct scaling regime for the analytic continuation.

9 Localisation and overlap: a discrete Witten surrogate

The metastable reduction target concerns the dynamics on packet masses once localisation has formed. A complementary diagnostic is to exhibit, in a controlled discrete semiclassical model, that the three well potential produces three localised low energy states whose overlaps define a mixing map between a natural energy eigenbasis and a natural corner localised basis.

9.1 A graph semiclassical operator on the simplex

Let A be the undirected adjacency matrix of the simplex lattice $\Delta_2^{(N)}$ and let $L = D - A$ be the unnormalised graph Laplacian, with D the degree matrix. For parameters $\kappa > 0$ and $\beta > 0$ define the discrete operator

$$H := \kappa L + \beta \operatorname{diag}(V), \tag{9.1}$$

where V is the restriction of **??** to $\Delta_2^{(N)}$.

In the regime of small κ and large β, the bottom of the spectrum of H concentrates near the minima of V, which are the three vertices of the simplex. In the symmetric case $\delta = 0$, the lowest band is expected to be (nearly) threefold degenerate for large enough N, with three eigenvectors spanning the space of corner localised states.

9.2 Corner basis, overlap matrix, and reduced operator

Let V_{eig} be the matrix of the lowest three eigenvectors of H (columns), and let U be the corner localised basis obtained by projecting the corner indicators χ_i onto $\operatorname{span}(V_{\mathrm{eig}})$ and orthonormalising. Define the overlap matrix

$$M := V_{\mathrm{eig}}^{\top} U. \tag{9.2}$$

This 3×3 matrix is the change of basis between an energy eigenbasis and a corner localised basis. In a physical continuation, an analogous overlap structure is the correct place for mixing information to live, since it is determined by localisation structure and the internal operator spectrum rather than by global index data.

Define also the reduced operator in the corner basis,

$$H_{\mathrm{slow}} := U^{\top} H U. \tag{9.3}$$

In the symmetric triple degeneracy regime, H_{slow} is close to a scalar multiple of the identity and the nontrivial structure is carried by M.

> *Remark* (Observed threefold degeneracy and a concrete overlap matrix). For a representative run with $N = 70$ (so $|\Delta_2^{(N)}| = 2556$ nodes), $\kappa = 10^{-4}$, $\beta = 200$, $\delta = 0$, and a corner radius $r = 0.06$, the lowest three eigenvalues of Eq. (9.1) are numerically equal to within solver tolerance:
>
> $$\lambda_0 \approx 1.9999289880\times10^{-4}, \qquad \lambda_1 \approx 1.9999289881\times10^{-4}, \qquad \lambda_2 \approx 1.9999289881\times10^{-4}, \tag{9.4}$$
>
> and the corner localised basis U has essentially unit localisation fractions on the three corner regions.

In the same run the overlap matrix Eq. (9.2) is, numerically,

$$M \approx \begin{pmatrix} 0.552563 & -0.629479 & -0.546288 \\ -0.749232 & -0.662287 & 0.005305 \\ -0.365139 & 0.406365 & -0.837581 \end{pmatrix}. \tag{9.5}$$

This is an explicit, reproducible example of how a three well internal landscape produces a three dimensional low energy sector with a nontrivial overlap between an energy eigenbasis and a corner localised basis.

Remark. The operator H is a surrogate diagnostic. It is not asserted to be the physical twisted Dirac operator or the physical $\mathcal{K}$ generator. Its role is to demonstrate, in a controlled finite model, that the CP2 moment potential produces three localised low energy states and a natural overlap matrix that can carry mixing information without contradicting the rigid global index constraint of Theorem 5.1.

10 Roadmap for further work

Theorem 5.1 already enforces the essential approach: within the globally honest hypercharge packaging, any three family mechanism must be local and dynamical. The work above isolates two theorem level bridges that would complete the replication story without weakening the index core.

10.1 Metastable reduction theorem for $\mathcal{K}$ on CP^2

A first theorem should take the schematic form stated in Remark 8.2, but for the genuine infinite dimensional generator $\mathcal{K}$ acting on densities on CP^2 (or on the full internal fibre Y with CP^1 spectator factor).

The analytic structure suggested by the simplex diagnostic is the following.

Fix a small parameter $\varepsilon > 0$ entering the Fisher regularisation and a symmetric three well potential $V(\mu)$ with barrier $\Delta V = 1/4$. Assume a local uniformly elliptic Fisher mobility and a well posed decomposition $\mathcal{K} = G + J$ with G generating a reversible gradient component and J preserving the stationary state.

Then show:

- The stationary state concentrates near the three fixed points as $\varepsilon \to 0$ and the transition rates between wells are exponentially small with exponent governed by the barrier $\Delta V = 1/4$.
- The spectrum of $\mathcal{K}$ near 0 has dimension three (one stationary mode and a complex conjugate pair) separated by a gap from the rest.
- There exist packet neighbourhoods U_i and a packet projection map $\rho \mapsto p$ such that, after an initial transient, $p(t)$ satisfies a closed effective dynamics $dp/dt = K_{\mathrm{slow}}(\varepsilon)p$ up to small corrections.
- The reduced operator inherits a weighted $G + J$ split and, in the symmetric case,

satisfies $\Pi_{\text{slow}} = (1/3, 1/3, 1/3)$.

The simplex diagnostic provides explicit, reproducible numerical targets for each bullet, including the barrier exponent and the symmetry class of K_{slow}.

10.2 Localisation and overlap theorem for twisted Dirac modes

A second theorem should address replication and mixing at the level of modes of the relevant internal operators. In the present note, the rigid index is computed for a global twisting E_{SM} and cannot be multiplied by global flux. The natural place for replication is therefore the existence of multiple localised copies of the same internal content in the three well landscape.

The correct theorem level claim to aim for is:

- In the symmetric three well regime, the relevant twisted operator (for example, a Witten deformed Dirac type operator compatible with the Fisher geometry) has a three dimensional low energy sector spanned by three localised modes supported near the three fixed points.
- These modes carry the same representation content, so replication is realised by localisation, not by global index multiplication.
- The mixing data is encoded in overlap matrices between a natural eigenbasis and a natural corner localised basis, and is therefore controlled by geometry and barrier penetration rather than by ad hoc parameters.

The discrete Witten surrogate of Section 10 is a controlled demonstration that this structure is not artificial: for the CP2 moment three well it is the generic semiclassical behaviour.

11 Conclusion

This note isolates a rigid net chiral index computation on the minimal internal fibre $Y = \mathrm{CP}^2 \times \mathrm{CP}^1$ under a globally honest hypercharge packaging. The resulting polynomial $I_{\text{SM}}(a, b, n_3)$ is explicit, and in the minimal branch the lepton indices are fixed while the colour sector depends affinely on the colour instanton number n_3.

A narrow but important obstruction is also proved: lepton family replication cannot be achieved by tuning a global hypercharge flux while keeping the line bundle twisting globally honest. This sets the correct scope boundary. Any three family mechanism in this framework must be local and dynamical.

The note then records a canonical three well Fisher sector native to CP^2 and states the corresponding metastable qutrit slow sector target $dp/dt = K_{\text{slow}} p$ with $K_{\text{slow}} = G_{\text{slow}} + J_{\text{slow}}$. Disciplined simplex diagnostics are provided to pin down the correct barrier height $\Delta V = 1/4$ and to exhibit explicit, reproducible examples of K_{slow} and its $G + J$ split in the appropriate symmetry class.

Finally, a discrete Witten surrogate is recorded to demonstrate that a three well CP2 moment potential naturally produces a three dimensional low energy sector with a nontrivial overlap matrix between an energy eigenbasis and a corner localised basis.

This is the correct mathematical location for mixing information to live, compatible with the rigid index core.

A Quick reference: key formulae

For convenience we list the main explicit polynomials and the canonical three well potential.

A.1 Index polynomials

$$I_{\mathrm{SM}}(a, b, n_3) = -48a^2 b + 60a^2 + 180ab - 4bn_3 - 4n_3 + 15. \tag{A.1}$$

$$I_L(a, b) = -27a^2 b + 9a^2 + 27ab - 9a - 6b + 2, \tag{A.2}$$

$$I_E(a, b) = -108a^2 b + 18a^2 + 54ab - 9a - 6b + 1. \tag{A.3}$$

A.2 Canonical three well potential on the moment simplex

$$V(\mu) = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, \qquad \Delta V = \frac{1}{4}. \tag{A.4}$$

B Reproducibility note (informal)

The discrete simplex diagnostics described in Sections 8 and 9 were implemented in two scripts, a qutrit metastability suite that constructs a reversible generator with a divergence free boundary circulation and extracts a reduced K_{slow}, and a localisation demo that constructs the graph operator $H = \kappa L + \beta\,\mathrm{diag}(V)$ and extracts an overlap matrix between an eigenbasis and a corner basis.

They are included only to set the correct barrier and symmetry class for subsequent analytic work. They are not asserted as physical models.